ECE 586BH: Problem Set 3
Fictitious play dynamics, minimum regret learning from forecasters

Due: Tuesday, October 16, at beginning of class
Reading: Course notes, Chapter 3

1. **[On potential games with two players]**
   (a) Consider a two player normal form game such that for some $(0,0) \in S \triangleq S_1 \times S_2$, the following holds for all $(a,b) \in S$:

   $$u_1(a,0) - u_1(0,0) + u_2(a,b) - u_2(a,0) = u_2(0,b) - u_2(0,0) + u_1(a,b) - u_1(0,b).$$

   Does the game necessarily have a potential function?

   (b) Consider a two-person zero sum game such that $u_2(a,b) = \ell(a,b)$ for all $(a,b) \in S$. Thus, $u_1(a,b) = -\ell(a,b)$. Express in terms of ℓ what it means for the game to be a potential game. Simplify as much as possible.

2. **[Best response dynamics for a three person game]**
   Consider the normal form game with $I = \{1,2,3\}$, $S_i = \{0,1\}$ for each i, and $u_i(s_1,s_2,s_3) = 1$ if the action s_i of player i is different from the actions of both of the other players, and $u_i(s_1,s_2,s_3) = 0$ otherwise. For brevity, we specify a mixed strategy for a single player i by a number $p_i \in [0,1]$, that indicates the probability the player selects action 1.

   (a) Find the best response set $B(p,q)$ for a player, which is the set of mixed strategies that maximize the (expected) payoff of the player, given the other two players are independently using the mixed strategies p and q, respectively.

   (b) Identify all Nash equilibria in mixed strategies, keeping in mind that mixed strategies include pure strategies as special cases.

   (c) Identify all trembling hand perfect equilibria.

   (d) Suppose the players use synchronous best response dynamics. To be definite, assume that if either action has the same payoff for a player against the strategies of the other two players in the previous round, the player selects by flip of a fair coin. Determine the long run limiting payoff per round for each player.

   (e) Is this a potential game? Justify your answer.

3. **[Regularized fictitious play for a three player game]**
   For the game of the previous problem, find and simplify as much as possible the equations for regularized fictitious play, including the ordinary differential equation, by modifying the equations from Section 3.4 of the notes. (Suppose for each player that the other two players are selecting actions independently according to their current state distributions.) The differential equation should involve the parameter $\tau > 0$. Numerically identify the equilibrium points and identify which ones are asymptotically stable, for $\tau = 1$ and for $\tau = 0.1$. A bit of Python code that could save you some time is available in a Python notebook file available at http:/courses.engr.illinois.edu/ece586GT/fa2018/pointer.html.

4. **[Hannan consistent estimators for a three player game]**
Consider further the game of the previous two problems.

   (a) Suppose player 1 uses a Hannan consistent estimator, where the two experts correspond to actions 0 and 1 for the player. What is the minimum guaranteed long-term payoff of the player, assuming arbitrary behavior of the other two players.

   (b) Suppose all three players use Hannan consistent estimators. Show that the long term sum of payoffs of players 1 through 3 per time slot is greater than or equal to 3/4. (Hint: There are two experts per player and Hannan consistency means each player's average payoff per unit time is greater than or equal to the payoffs of both of his/her experts.)

5. **[Strong law of large numbers for discrete-time martingales with bounded increments]**
Show that if $(D_t : t \geq 0)$ is a martingale difference sequence such that $|D_t| \leq 1$ for all t, then the corresponding martingale defined by $X_n = D_1 + \cdots + D_n$ satisfies $\limsup_{n\to\infty} \frac{X_n}{2\sqrt{n \ln n}} \leq 1$ almost surely (i.e. with probability one). (Hint: Use the Azuma-Hoeffding inequality and the Borel Cantelli lemma – both in the notes.) In particular, $\frac{X_n}{n} \to 0$ almost surely. (This is used to help establish the existence of Hannan consistent estimators for players of a finite game. See the paragraph following Corollary 3.26.

6. **[Blackwell approachability for a simple game with vector valued payoff]**
Consider a two player game with strategy spaces

$$S_1 = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\} \text{ and } S_2 = \left\{ \left(\begin{array}{c} 1 \\ 1 \end{array} \right), \left(\begin{array}{c} -1 \\ -1 \end{array} \right), \right\},$$

such that if player 1 selects $x \in S_1$ and player 2 selects $y \in S_2$, the payoff vector for player 1 is $x + y$.

   (a) An arbitrary half space H in the plane can be expressed as $H = \{v : v \cdot u \leq c\}$, where u is a unit length vector in $\mathbb{R}^2$ and $c \in \mathbb{R}$. Under what condition on u and c is this half space approachable (for player 1)?

   (b) Let $B(r)$ denote the disk in $\mathbb{R}^2$ of radius r. Find the minimum value of r so that $B(r)$ is approachable (for player 1).