University of Illinois **Fall 2018**

ECE 586GT: Problem Set 2

Uniqueness of Nash equilibria, zero sum games, evolutionary dynamics

Due: Tuesday, Sept. 25, at beginning of class
Reading: Course notes, Sections 1.5-1.7 and Chapter 2

1. [**Existence and uniqueness of NE for games with continuous type strategies**]
   This problem concerns an n player game with strategy space $S_i = (0, +\infty)$ for all players i, and space of strategy profiles $S = S_1 \times \cdots \times S_n$.

   (a) Consider the payoff functions $u_i(s) = (\ln s_i) - s_i s_{\mathsf{tot}}$, where $s_{\mathsf{tot}} = s_1 + \cdots + s_n$. Does there exist a pure strategy Nash equilibrium? If so, is it unique? Can you find an explicit expression for it?

   (b) Consider the payoff functions $u_i(s) = (\ln s_i) - s_i s_{\mathsf{tot}} + \epsilon \sum_j \cos(s_j)$ for some $\epsilon > 0$. Find a positive constant $\overline{\epsilon}$ such that there exists a Nash equilibrium if $0 \leq \epsilon < \overline{\epsilon}$. To avoid some tedious steps, you may assume that the strategy space for each player is restricted to the closed interval $[\delta, 1/\delta]$, such that δ can be chosen so small that any best response for any player is in the interior of the interval.

   (c) For the payoff functions in part (b), find a positive constant $\overline{\epsilon}$ such that there exists a *unique* pure strategy Nash equilibrium if $0 \leq \epsilon < \overline{\epsilon}$. (Hint: A diagonal matrix with negative entries on the diagonal is negative definite, and the sum of a negative definite matrix and a negative semi-definite matrix is negative definite.)

2. [**Market power in territory control game**]
   Recall the territory control game for a graph $G = (V, E)$ and $n \geq 2$ players, defined in problem set 1. The game is symmetric in the sense that the payoff functions are invariant under permutation of the players. A strategy profile (in either pure or mixed strategies) is symmetric if every player uses the same strategy.

   (a) Prove that for any symmetric finite player game, there exists a symmetric Nash equilibrium in mixed strategies. Thus, for the territory control game based on a connected graph $G = (V, E)$, there is a Nash equilibrium such that the expected payoff of each player is $\frac{|V|}{n}$.

   (b) Show that a two-player game such that the sum of payoffs is constant for all strategy pairs is equivalent to a zero sum two-player game. In particular, there exists a saddle point and a value of the game for each player.

   (c) Suppose there are three players, but players 2 and 3 team up and split their payoffs with each other. Thus, players 2 and 3 together can be viewed as a super player; from the perspective of player 1, the game becomes a two-player game, with constant sum of payoffs $|V|$. Player 1 could be placing a Burger King and players 2 and 3 together place two McDonalds. Show that the value of the modified game for player 1 is greater than or equal to $|V|/4$ for any connected graph G. (Hint: What if there were two super players, each selecting nodes for two restaurants?)

   (d) Consider the scenario of part (c) for the special case of a line graph $V = \{1, 2, \ldots, m\}$, such that $m = 4b + 2$ for some nonnegative integer b, and such that $E = \{[i, i+1] : 1 \leq$

$i \le m-1\}$. Show that the value of the modified game for player 1 is equal to $|V|/4$. (Hint: By part(c) only an upper bound on the average payoff of player 1 is needed.)

3. **[Dual of a transport problem]**
Let $n \ge 1$ and let W be an $n \times n$ matrix with nonnegative elements. Consider the following linear optimization problem:

$$\begin{aligned} &\min_X \sum_{i,j\in[n]} X_{i,j}W_{i,j} \\ &\text{subject to:} \quad \sum_{j\in[n]} X_{i,j} = 1 \qquad \text{for } i \in [n] \\ &\qquad\qquad\quad \sum_{i\in[n]} X_{i,j} = 1 \qquad \text{for } j \in [n] \\ &\qquad\qquad\quad X_{i,j} \ge 0 \qquad \text{for } i,j \in [n] \end{aligned}$$

An interpretation is that there are n sources of some good, n destinations, $W_{i,j}$ is the cost of transporting a unit of good from i to j, and $X_{i,j}$ is the amount of good transported from i to j. Each source provides one unit of good and each destination receives one unit of good. The problem in vector matrix form can be written as:

$$\begin{aligned} &\min_X \ \langle X, W\rangle \\ &\text{subject to: } X\mathbf{1} = \mathbf{1},\ X^T\mathbf{1} = \mathbf{1},\ X \ge 0. \end{aligned}$$

(a) Derive the dual problem by first finding the Lagrangian function. Find a simple version of the dual problem. (Hint: You can either eliminate the Lagrange multipliers for the constraints $X_{i,j} \ge 0$, or don't handle those constraints with Lagrange multipliers in the first place.

(b) Find the common value of the primal and dual, and solutions, for $W = \begin{pmatrix} 3 & 8 \\ 2 & 4 \end{pmatrix}$.

4. **[Evolutionarily stable strategies and states]**

Consider the following symmetric, two-player game::

	1	2
1	1,1	1,2
2	2,1	0,0

(a) Does either player have a (weakly or strongly) dominant strategy?

(b) Identify all the pure strategy and mixed strategy Nash equilibria.

(c) Identify all evolutionarily stable pure strategies and all evolutionarily stable mixed strategies.

(d) The replicator dynamics based on this game represents a large population consisting of type 1 and type 2 individuals. Show that the evolution of the population share vector $\theta(t)$ under the replicator dynamics for this model reduces to a one dimensional ordinary differential equation for $\theta_t(1)$, the fraction of the population that is type 1.

(e) Identify the steady states of the replicator dynamics.

(f) Of the steady states identified in the previous part, which are asymptotically stable states of the replicator dynamics? Justify your answer.