University of Illinois **Fall 2018**

ECE 586GT: Problem Set 1
Analysis of static games

Due: Tuesday, Sept. 11, at beginning of class
Reading: Course notes, Sections 1.1 - 1.4

1. [**A random zero sum game**]
   Consider a two-player zero-sum game corresponding to the matrix $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. The players select an element of the matrix in the following way: player 1 selects a row and player 2 selects a column. The selected element is the payoff of player 2 and the negative of the payoff of player 1. Thus, player 1 seeks to minimize the element selected and player 2 seeks to maximize it. Suppose that A, B, C, D are independent and identically distributed continuous-type random variables. Suppose both players know the values of the random variables before selecting their actions. (Hint: The event two or more of the random variables are equal has probability zero, so assume without loss of generality that the random variables take different values. By symmetry, the probability any k of the random variables have a particular order is $1/k!$. For example, $\mathbb{P}\{A < B < C\} = 1/6$.)

   (a) Find the probability there is more than one pure-strategy Nash equilibrium (NE).
   (b) Find the probability there is at least one pure-strategy NE.
   (c) Find the probability player 1 has a dominant strategy.
   (d) Find the probability both players have a dominant strategy.

2. [**Territory control game on a graph**]
   An application of the game in this problem could be for each of a set of players to decide where to place their fast food restaurant, under the assumption any customers will travel to the nearest fast food restaurant. Suppose $n \geq 2$ and $G = (V, E)$ is an undirected graph with vertex set V and edge set E. Consider the n player game such that the action of each player is to select a vertex in V. Two or more players can select the same vertex. Given the set of vertices selected by the players, each vertex v in the graph assigns total payoff one among the players, divided equally among those players who selected vertices the closest to the vertex v, where the distance between two vertices is the minimum number of edges in a path from one vertex to the other. For example, if there are no ties, the vertex v assigns payoff one to the player with selected vertex closest to v. The total payoff of a player is the sum over all vertices of the payoff assigned to that player. Thus, the sum of the payoffs of all players is $|V|$ for any choices of the players.

   (a) Does there always exist at least one Nash equilibrium in mixed strategies?
   (b) Consider the line graph G with $V = \{1, 2, \ldots, 100\}$ and $E = \{[i, i+1] : 1 \leq i \leq 99\}$. Suppose there are two players ($n = 2$). Find the set of all pure strategy Nash equilibria.
   (c) Is there a dominant strategy for a player in the game of part (b)?
   (d) Repeat part (b) for three players, $n = 3$.

3. **[Possession is nine-tenths of the law]**
Consider the normal form game $G = (I, (S_i), (u_i))$ with $I = \{1, \ldots, n\} = [n]$, $S_i = [0, 1]$, and $u_i(s_i) = \begin{cases} s_i & \text{if } s_1 + \ldots + s_n \leq 1 \\ 0 & \text{else} \end{cases}$. For example, suppose there is a pie, and every player declares what fraction of the pie he/she will take. Players get what they declare if the sum of the fractions is less than or equal to one. Else none get any pie.

   (a) Find all dominant strategies for a given player. Justify your answer.

   (b) Find all Nash equilibria in pure strategies. (Include them all. Even ones such that all players get zero payoff.)

4. **[Agreeing to disagree]**
Suppose $n \geq 3$ and consider the normal form game with n players, $I = [n]$, placed in a circle in order of index with wrap-around, so player n is next to players $n-1$ and 1. Let $S_i = \{0, 1\}$; each player i declares a bit s_i. Let $u_i(s_i) = \begin{cases} 1 & \text{if } s_{i-1} = s_{i+1} \neq s_i \\ 0 & \text{else} \end{cases}$ for $i \in I$, where, by notational convention, $s_0 = s_n$ and $s_{n+1} = s_1$.

   (a) Find a simple rule to determine whether a given strategy profile $s = (s_1, \ldots, s_n)$ is a Nash equilibrium. Justify your answer.

   (b) Give an example of a Nash equilibrim in non-degenerate mixed strategies. Justify your answer.

5. **[True or false]**
Show the following statement is true, or show it is false. If a player in a normal form finite game has a dominant strategy, then the player must play that strategy with probability one for any correlated equilibrium.

6. **[Guessing 2/3 of the average]**
Consider the following game for n players. Each of the players selects a number from the set $\{1, \ldots, 100\}$, and a cash prize is split evenly among the players who's numbers are closest to two-thirds the average of the n numbers chosen.

   (a) Show that the problem is solvable by iterated elimination of *weakly* dominated strategies, meaning the method can be used to eliminate all but one strategy for each player, which necessarily gives a Nash equilibrium. (A strategy μ_i of a player i is called weakly dominated if there is another strategy μ_i' that always does at least as well as μ_i, and is strictly better than μ_i for some vector of strategies of the other players.)

   (b) Give an example of a two player game, with two possible actions for each player, such that iterated elimination of weakly dominated strategies can eliminate a Nash equilibrium. (Hint: The eliminated Nash equilibrium might not be very good for either player.)

   (c) Show that the Nash equilibrium found in part (a) is the unique mixed strategy Nash equilibrium (as usual we consider pure strategies to be special cases of mixed strategies). (Hint: Let k^* be the largest integer such that there exists at least one player choosing k^* with strictly positive probability. Show that $k^* = 1$.)