University of Illinois Fall 2018

ECE 586GT: Exam I

Thursday, October 18, 2018
7:00 p.m. — 8:30 p.m.
2013 Electrical Engineering Building

1. **[42 points]** Consider the two-player zero sum game in which player one, the row selector, seeks to minimize ℓ and player 2, the column selector, seeks to maximize ℓ, where ℓ is given by the following table:

		Player 2		
		1	2	3
	1	0	1	2
Player 1	2	1	0	1
	3	2	1	0

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As usual, pure strategies are special cases of mixed strategies.

(a) (6 points) Identify all minmax optimal pure strategies for player 1.
Solution: Since $\max_j \ell(1,j) = 2, \max_j \ell(2,j) = 1, \max_j \ell(3,j) = 2$, action 2 is the unique minmax optimal pure strategy for player 1.

(b) (6 points) Identify all maxmin optimal pure strategies for player 2.
Solution: Since $\min_i \ell(i,j)$ is zero for any j, all three actions 1,2,3 are maxmin optimal pure strategies for player 2.

(c) (12 points) Identify all maxmin optimal mixed strategies for player 2.
Solution: In other words, find the mixed strategies q for player 2 that maximize $\min\{q_2 + 2q_3, q_1 + q_3, 2q_1 + q_2\}$. This minimum is less than or equal to $q_1 + q_3$ which is less than or equal to one. If q can be selected to make the minimum equal to one it must be optimal. Making the minimum one requires $q_2 = 0$, in which case the problem is to select q_1, q_3 to maximize $\min\{2q_3, 1, 2q_1\}$, which is achieved uniquely by $q = (1/2, 0, 1/2)$. Another approach is to use the fact, for the optimal q, at least two terms inside the minimization should be equal, and the third term greater than or equal to the other two.

(d) (12 points) Identify all minmax optimal mixed strategies for player 1.
Solution: In other words, find the mixed strategies p for player 1 that minimizes $\max\{p_2 + 2p_3, p_1 + p_3, 2p_1 + p_2\}$. At any solution at least two of the terms should be equal and the third term less than or equal to the first two. Setting the first and third terms equal yields $p_2 + 2p_3 = 2p_1 + p_2$ or $p_1 = p_3$, or $p_1 = p_3 = (1 - p_2)/2$. Expressing the maximum in terms of p_2 yields $\max\{1, 1 - p_2, 1\}$, so that any mixed strategy for player 1 with $p_1 = p_3$ is minmax optimal. An attempt to find more solutions is to set the first two terms equal: $p_2 + 2p_3 = p_1 + p_3$ or $p_1 = p_2 + p_3$. It follows that $p_1 = 1/2$. Using $p_3 = 1/2 - p_2$, the max loss is $\max\{1 - p_2, 1 - p_2, 1 + p_2\}$, which doesn't have the two terms being larger than the third unless $p_2 = 0$, so no new solutions are found this way. By symmetry, no new solutions are found by setting the second and third terms to be equal. In summary, p is minmax optimal if and only if $p_1 = p_3$.

(e) (6 points) Identify the maximum expected payoff to player 2 over all correlated equilibria (Hint: Can be done with no calculation beyond what you did for parts (c) or (d).)
Solution: For two person zero sum games, the value profile for any correlated Nash equilibrium is equal to the value of the game for Nash equilibria. That is because each player can do no worse than achieve the value of the game by ignoring the oracle and playing the Nash equilibrium strategies. So the maximum expected payoff to player 2 over all correlated equilibria is 1.

2. **[30 points]** Consensus games (Coles and Olives, 1980). Consider a normal form game with finite set of players I and action sets $S_i = \{0, 1\}$ for all $i \in I$. For each player i let A_i, with $A_i \subset I \backslash \{i\}$. For a strategy profile $s = (s_i)_{i \in I}$, let $u_i(s) = \sum_{j \in A_i} \mathbf{1}_{\{s_i = s_j\}}$. In other words, the payoff of player i is the number of players in A_i with which the player agrees.

   (a) (15 points) Suppose the sets $(A_i)_{i \in I}$ are neighborhood sets for an undirected graph. In other words, suppose for each $i, j \in I$, $i \in A_j$ if and only if $j \in A_i$. Is the game necessarily a potential game? If so, identify the potential function. If not, give an example and argue why a potential function does not exist for it.
   Solution: Yes. Call i and j neighbors if $i \in A(j)$. Let $\Phi(s)$ be the number of pairs of neighbors $\{i, j\}$ such that $s_i = s_j$. To see why Φ is a potential, note that for any player i, $\Phi(s) = u_i(s) + k_i(s)$, where $k_i(s)$ does not depend on s_i, namely, $k_i(s)$ is the number of pairs of neighbors $\{j, k\} \subset I \backslash \{i\}$ such that $s_j = s_k$.

   (b) (15 points) Repeat part (a), but without the assumption $i \in A_j$ if and only if $j \in A_i$.
   Solution: The game does not have to be a potential game. Suppose, for example, that $I = \{1, 2\}$, $A_1 = \{2\}$ and $A_2 = \emptyset$. If Φ were a potential function, then without loss of generality we can assume $\Phi(0, 0) = 0$. Then $\Phi(0, 1) = 0$, so $\Phi(1, 1) = 1$, so $\Phi(1, 0) = 1$, so $\Phi(0, 0) = 2$. A contradiction. Hence, there is no potential. (Similarly, the game is not a potential game for any graph with two vertices i, j such that $j \in A(i)$ by $i \notin A(j)$.

3. **[28 points]** Determine whether each statement is TRUE or FALSE, *and give a justification for your answer* for more than half credit.

   (a) $x = 0.5$ is a stable equilibrium for the differential equation $\dot{x}_t = x_t(1 - x_t)(x_t - 0.5)^2$.
   Solution: False. For any $\epsilon > 0$ with $\epsilon < 1/4$, if the initial state is $0.5 + \epsilon$, then the righthand side is bounded below by a strictly positive number δ over the interval $[0.5 + \epsilon, 1 - \epsilon]$, so starting from $0.5 + \epsilon$, the trajectory will reach $1 - \epsilon$ before time $0.5/\delta$, and in particular it will not stay close to 0.5. So 0.5 is not a stable point.

   (b) Suppose the sequence of strategy profiles produced by iterated best response for some finite normal form game is periodic with period 2, alternating between strategy profile vectors $s^{(0)}$ and $s^{(1)}$ in $S = S_1 \times \cdots \times S_n$. Let s^* be the mixed strategy profile such that, $s_i^* = s_i^{(0)}$ with probability one half, and $s_i^* = s_i^{(1)}$ with probability one half, with the choices being made independently for different players. Then s^* must be a Nash equilibrium vector.
   Solution: False. Consider for example the coordination game with payoff matrix

		Player 2	
		1	2
Player 1	1	1,1	0,0
	2	0,0	2,2

   It could be $s^{(0)} = (1, 2)$ and $s^{(1)} = (2, 1)$. Then for s^*, each player selects and action by fair coin flip. That is not a Nash equilbrium.

   (c) Consider a two player game with vector valued payoffs and suppose sets S_1 and S_2 are both approachable by player 1. Then $S_1 \cup S_2$ must be approachable by player 1.
   Solution: True. If L_t represents the time average of the payoffs of player 1 up to time t, player 1 can ensure that $d(L_t, S_1) \to 0$ with probability one. Since $d(L_t, S_1 \cup S_2) \leq d(L_t, S_1) \to 0$, it follows that $d(L_t, S_1 \cup S_2) \to 0$ as well.

   (d) Consider a two player game with vector valued payoffs and suppose sets S_1 and S_2 are both approachable by player 1. Then $S_1 \cap S_2$ must be approachable by player 1.
   Solution: False. There are many simple counter examples. The following is simple but somewhat degenerate. Consider a one player game such that in each round player 1 can select payoff 1 or payoff -1. Sets $\{1\}$ and $\{-1\}$ are approachable for player 1, but $\{1\} \cap \{-1\} = \emptyset$ is not.