University of Illinois **Fall 2017**

ECE 586BH: Problem Set 6: Problems and Solutions
Optimal selling mechanisms, and cooperative games

Due: Thursday, December 7, at beginning of class
Reading: Course notes, Sections 6.2 and Chapter 7. Also recommended: V. Krishna, *Auction Theory,* Chapter 5 (based largely on Myerson(1982)) and Chapter 6 (based on Milgrom and Weber(1983))

1. **[Revenue optimal auction with identically distributed uniform valuations]** Consider the design of a mechanism to sell a single object to maximize the expected payoff to the seller, assuming the value of the object to the seller is $r = 0$ (free disposal) and there are n bidders with $n \geq 1$ such that the valuations $(X_i)_{1 \leq i \leq n}$ are independent, and each uniformly distributed over the interval $[a, b]$, where $0 \leq a < b$.

   (a) Identify the IC, IR selling mechanism that maximizes the expected payoff of the seller, assuming the min to win payment rule is used. Be as explicit as possible.
   Solution: For the uniform distribution over $[a, b]$, $1 - F(x) = \frac{b-x}{b-a}$ and $f(x) = \frac{1}{b-a}$ for $x \in [a, b]$, so $\psi_i(x_i) = 2x_i - b$ for $x_i \in [a, b]$. Note that ψ_i is increasing in x_i so the design problem is proper, and $\psi(x_i) \geq 0$ is equivalent to $x_i \geq \frac{b}{2}$. So the allocation rule is to give the object to the highest bidder if the highest bid is at least $\frac{b}{2}$, and, otherwise, not give the object to any bidder. A bidder that doesn't get the object pays zero, and if a bidder gets the object then the payment of the bidder is the maximum of $\frac{b}{2}$ and the second highest bid. Thus, the allocation mechanism is the Vickrey second price action with reserve price $\frac{b}{2}$.

   (b) Find the expected payoff of the seller in case of a single bidder, $n = 1$. (Hint: Treat the cases $a < \frac{b}{2}$ and $a \geq \frac{b}{2}$ separately.)
   Solution: (i) If $a < \frac{b}{2}$ then

   $$m_1(X_1) = \begin{cases} \frac{b}{2} & \text{if } \frac{b}{2} \leq X_1 \leq b \\ 0 & \text{else} \end{cases},$$

   so $\mathbb{E}[m_1(X)] = \frac{b}{2}\mathbb{P}\left\{\frac{b}{2} \leq X_1 \leq b\right\} = \frac{b^2}{4(b-a)}$.
   (ii) If $a \geq \frac{b}{2}$ then $\mathbb{P}\left\{X_1 \geq \frac{b}{2}\right\} = 1$, and thus $m(X_1) = a$ with probability one. In other words, the seller charges price a and the bidder accepts it with probability one. The expected payoff of the seller is a.

2. **[Optimal seller mechanism for single minded bidders]** Suppose a seller wishes to sell a finite collection of objects $\mathcal{O}$, and the value to the seller of any subset of objects is zero (i.e. free disposal) so the payoff of the seller is the sum of payments made by the bidders. Suppose there are n bidders indexed by I. Suppose each bidder i has a random positive value X_i for some particular bundle $\hat{A}_i$ and value zero for any other bundle. Suppose the random variable X_i is distributed over some interval $[0, w_i]$ with positive density f_i and virtual valuation function ψ_i. Suppose ψ_i is strictly increasing for each i, so the design problem is proper. Suppose the sets $(\hat{A}_i)$ and the distributions of the values are known to the seller and all bidders. Describe an IC, IR selling mechanism that maximizes the expected seller revenue.

Solution: Each bidder i submits a nonnegative bid $\hat{X}_i$. The allocation mechanism used by the seller determines for each bidder i a function $q_i(x_i)$, which is the probability the bidder is allocated the object given the bidder submitted bid x_i. Revenue equivalence holds as in the case of the sale of a single object, because from the standpoint of any bidder, the game appears as a single object sale. Consequently, the expected payment function, $m_i(x_i)$, for the revenue optimal auction is completely determined by the function q_i. Seller thus seeks to use a selection rule to solve the following optimization problem:

$$\begin{aligned} &\max \mathbb{E}\left[\sum_{i \in I} \psi_i(X_i) \mathbf{1}_{\{i \text{ gets bundle } \hat{A}_i\}}\right] \qquad (1) \\ &\text{with respect to } Q \\ &\text{subject to } x_i \mapsto q_i(x_i) \text{ nondecreasing for each } i \end{aligned}$$

Due to the assumption that the ψ_i's are increasing, the constraint that the allocation probability functions q_i be nondecreasing is not binding (in other words, it can be ignored). So the mechanism selects the set of disjoint bundles to maximize the sum of the virtual valuations of the bundles. The mechanism can also use the min to win payment rule, such that the payment of each bidder i that gets his/her bundle is the minimum bid the bidder would have needed use to have still gotten the bundle.

3. [**Optimal seller mechanism with discrete valuation distributions**]
Myerson's theory of optimal auctions can be developed for discrete valuation distributions with only minor modifications. Suppose there is a seller with a single object to sell and n bidders indexed by I. Let X_i denote the value of the object to bidder i. Assume the X_i's are independent and identically distributed with $\mathbb{P}\{X_i = x_k\} = p_k$ for $1 \leq k \leq K$ and $i \in I$, where $x_1 < \ldots < x_K$ and $p = (p_1, \ldots, p_K)$ is a probability vector with strictly positive probabilities. The revelation principle holds for discrete distributions with the same proof as given in the notes for the continuous setting. (We restricted attention to the case of identical distributions only to simplify the notation.)

   (a) Let q_k denote the probability a bidder gets the object given the bidder bids x_k and let m_k denote the expected payment the bidder makes given bid x_k. Adopt the notational convention that q and m can also be thought of as functions with $q(x_k) = q_k$ and $m(x_k) = m_k$ for $1 \leq k \leq K$. Show that for a given q, it is possible to select m to satisfy the IC constraint if and only if q_k is nondecreasing in k. Furthermore, show that if q_k is nondecreasing in k, then the maximum choice of the function m subject to the IC and IR constraints is given by (with $q_0 \triangleq 0$):

$$m_k = \sum_{j=1}^{k} (q_j - q_{j-1}) x_j \quad \text{for } 1 \leq k \leq K. \qquad (2)$$

   (Note: Unlike the case of continuous values, q and the IC and IR constraints do not unquely determine m, but there is still a unique pointwise maximum m, given by (2), satisfying the IR and IC constraints.)(Hint: Use induction on k. For the base case, why must m_1 satisfy $m_1 \leq q_1 x_1$?)

   Solution: (only if) Suppose the mechanism is IC for some choice of m. By the definition of IC, for $k, k' \in \{1, \ldots, K\}$,

$$q_k x_k - m_k \geq q_{k'} x_k - m_{k'} \tag{3}$$
$$q_{k'} x_{k'} - m_{k'} \geq q_k x_{k'} - m_k \tag{4}$$

Adding the respective sides of (3) and (4) and rearranging yields

$$(q_k - q_{k'})(x_k - x_{k'}) \geq 0,$$

showing q_k is nondecreasing in k.

(if) We shall prove the IC condition, (3), for all $k, k' \in \{1, \ldots, K\}$, assuming q_k is nondecreasing and m_k is given by (2). If $k' < k$ then

$$m_k - m_{k'} = \sum_{j=k'+1}^{k} (q_j - q_{j-1}) x_j \leq \sum_{j=k'+1}^{k} (q_j - q_{j-1}) x_k = (q_k - q_{k'}) x_k$$

and if $k' > k$ then

$$m_{k'} - m_k = \sum_{j=k+1}^{k'} (q_j - q_{j-1}) x_j \geq \sum_{j=k+1}^{k'} (q_j - q_{j-1}) x_k = (q_{k'} - q_k) x_k$$

Thus, $m_k - m_{k'} \leq (q_{k'} - q_k) x_k$, or, equivalently, (3) holds, for all $k, k' \in \{1, \ldots, K\}$, as claimed.

Finally, we show that (2) gives the maximum m subject to the IC and IR constraints. The IR constraint requires $m_1 \leq q_1 x_1$, because if the value of the object to the bidder is x_1, then the expected payoff of the bidder bidding x_1 is $q_1 x_1 - m_1$, which must be nonnegative. Equation (3) with $k' = k - 1$ can be rearranged to give $m_k - m_{k-1} \leq (q_k - q_{k-1}) x_k$. Therefore, by induction on k, any choice of m satisfying the IR and IC conditions must be less than or equal to m given by (2).

(b) Show there exists a virtual valuation function ψ so that

$$\mathbb{E}\left[m(X_i)\right] = \mathbb{E}\left[\psi(X_i) \mathbf{1}_{\{\text{bidder } i \text{ gets object}\}}\right].$$

(Hint: Use the notation $\psi(x_j) = \psi_j$ and find an expression for $\psi_1, \ldots, \psi_K$ in terms of the x's and p's.) The value distribution is said to be proper if ψ_j is increasing in j. Describe the optimal selling mechanism in a manner similar to the continuous case described in the notes, under the assumption the value distribution is proper. Let r denote the value of the object to seller in case it is not given to any bidder.

Solution:

$$
\begin{aligned}
E[m(X)] &= \sum_{k=1}^{K} p_k m_k \\
&= \sum_{j,k:1\leq j\leq k\leq K} (q_j - q_{j-1}) x_j p_k \\
&= \sum_{j=1}^{K} q_j \left(\sum_{k=j}^{K} x_j p_k - \sum_{k=j+1}^{K} x_{j+1} p_k \right) \\
&= \sum_{j=1}^{K} q_j \left(x_j p_j - (x_{j+1} - x_j) \sum_{k=j+1}^{K} p_k \right) \\
&= \sum_{j=1}^{K} p_j q_j \psi_j = \mathbb{E}\left[\psi(X_i) \mathbf{1}_{\{\text{bidder } i \text{ gets object}\}} \right].
\end{aligned}
$$

where $\psi_j \triangleq x_j - (x_{j+1} - x_j)\frac{(\sum_{k=j+1}^{K} p_k)}{p_j}$. The above is written with the understanding $\sum_{k=j+1}^{K}(\cdot) = 0$ if $j = K$ so that $\psi_K = x_K$. The expected payoff of the seller can thus be written as

$$
U_0 = \mathbb{E}\left[\sum_{i\in I} \psi(X_i) \mathbf{1}_{\{\text{bidder } i \text{ gets object}\}} + r \mathbf{1}_{\{\text{no bidder gets object}\}} \right]
$$

Thus, U_0 is maximized with respect to Q by the

MAX VIRTUAL VALUATION selection rule	• Select winner from $\arg\max_i \psi_i(X_i)$ if $\max_i \psi_i(X_i) \geq r$ • Select no winner if $\max_i \psi_i(X_i) < r$

Suppose the value distribution is proper, so that ψ^j is increasing in j. Then the above selection rule leads to q_j being nondecreasing in j, so it is the optimal selection rule subject to the IC and IR constraints. A detailed payment rule that can be used is:

MIN TO WIN payment rule: $M_i(x) = y_i(x_{-i}) \mathbf{1}_{\{i \text{ gets object}\}}$.

where, as in the continuous values case,

$$
y_i(x_{-i}) = \min\{z_i : \psi_i(z_i) \geq r \text{ and } \psi_i(z_i) \geq \psi_j(x_j) \text{ for } j \neq i\}.
$$

(c) Let $0 < L < H$ and $0 < p < 1$. Specialize the results of this problem to the case of a single bidder with a value X satisfying $\mathbb{P}\{X = H\} = p$ and $\mathbb{P}\{X = L\} = 1 - p$. Suppose the value of the object to seller, r, is zero (free disposal). Find the maximum expected revenue of the seller for IC and IR seller mechanisms. Calculate the expected value of the object to the bidder minus the expected revenue of the seller. This is called the *information rent*. It is the value the seller cannot extract because seller does not know the value of X.

Solution: Using the above formulas for $(p_1, p_2) = (p, 1-p)$ and $(x_1, x_2) = (L, H)$ yields $\psi_L = L - (H - L)\frac{p}{1-p} = \frac{L - Hp}{1-p}$ and $\psi_H = H$. We consider two cases:

Case 1: $L \geq pH$ Then $\psi_H \geq \psi_L \geq 0$, so bidder always gets the object and always pays L. In this case, the mean revenue to seller is L.

Case 2: $L < pH$ Then the bidder gets the object if he/she bids H and doesn't get the object if he/she bids L. Under the min to win payment rule, the bidder pays H if the bidder bids H and pays zero (and doesn't get the object) if the bidder bids L. In this case the mean revenue to seller is pH.

In general, the mean revenue to seller is $\max\{L, pH\}$. The expected value of the object to the bidder is $pH + (1-p)L$. Therefore, the information rent (in other words, the value that is not extracted by the seller) is given by:

$$pH + (1-p)L - \max\{L, pH\} = \begin{cases} p(H-L) & 0 \leq p \leq \frac{L}{H} \\ (1-p)L & \frac{L}{H} \leq p \leq 1. \end{cases}$$

4. **[Production economy with convex productivity function]**
Recall the production economy of Example 7.5 in the course notes. The set of players is $I = \{c\} \cup W$, where c represents a factory owner, and W is a set of m workers, for some $m \geq 1$. In order for a coalition to have positive worth, it must include the owner and at least one worker, because both the factory and at least one worker are needed for production. Suppose the worth of a coalition consisting of the owner and k workers is $f(k)$, where $f : [m] \to \mathbb{R}$ is such that $f(0) = 0$, f is monotone increasing, and its increments $f(k+1) - f(k)$ are nondecreasing in k over $0 \leq k \leq m-1$. (In contrast, in the notes, the increments are assumed to be nonincreasing.)

(a) Identify the core of the game. Simplify your answer as much as possible. Hint: Given $(x_c, x_1, \ldots, x_m)$, let $x_{[1]} \leq \ldots \leq x_{[m]}$ denote the ordered permutation of $x_1, \ldots, x_m$. Express your answer using $x_{[i]}$'s.)

Solution: As shown in the example, $(x_c, x_1, \ldots, x_m)$ is in the core if and only if

$$x_c + x_1 + \cdots + x_m = f(m) \tag{5}$$

$$x_c + \sum_{i \in S_w} x_i \geq f(k) \text{ if } |S_w| = k \tag{6}$$

$$x_i \geq 0 \quad i \in I = \{c\} \cup W \tag{7}$$

Given $(x_c, x_1, \ldots, x_m)$, let $x_{[1]} \leq \ldots \leq x_{[m]}$ denote the ordered permutation of $x_1, \ldots, x_m$ Constraint (6) is equivalent to $x_c + \sum_{j=1}^{k} x_{[j]} \geq f(k)$ for $1 \leq k \leq m$, and if (5) holds (6) is equivalent to $\sum_{i=k+1}^{m} x_{[i]} \geq f(m) - f(k)$ for $1 \leq k \leq m-1$. In words, the sum of the $m-k$ largest payoffs to workers must be less than or equal to the sum of the last (i.e. largest) $m-k$ increments of f. So the core is:

$$\{(x_c, x_1, \ldots, x_m) \in \mathbb{R}_+^{m+1} : \sum_{i=k+1}^{m} x_{[i]} \geq f(m) - f(k) \text{ for } 1 \leq k \leq m-1$$
$$\text{and } x_c + x_1 + \cdots + x_m = f(m)\}$$

(b) Find the Shapley payoff profile.

Solution: Given random order of arrival, the marginal value of c is $f(K)$, where K is the number of workers that arrive before c. Since K is uniformly distributed from 0 to m, $x_c^{Shapley} = \frac{1}{m+1}\sum_{k=0}^{m} f(k)$. The value of the grand coalition I is $f(m)$, so value $f(m) - x_c^{Shapley}$ is to be shared among the workers. By symmetry, $x_i^{Shapley}$ is the same for all workers. So $x_i^{Shapley} = \frac{1}{m}\left(f(m) - \frac{1}{m+1}\sum_{k=0}^{m} f(k)\right)$ for any worker i.

(c) Is the Shapley payoff profile in the core?

Solution: Yes. Can show directly, or notice that the valuation function v is supermodular– the marginal value of adding any player to a coalition is an increasing function of the coalition.

5. **[A market with two goods]**
Consider the market with transferrable utilities $M = (I, \ell, (w_i), (f_i))$ such that $I = \{1, 2, 3\}$, $\ell = 2$, and

$$
\begin{aligned}
f_1(z_{1,1}, z_{1,2}) &= (\min\{z_{1,1}, z_{1,2}\})^{1/2} & w_1 &= (0, 0)\\
f_2(z_{2,1}, z_{2,2}) &= (0.1)(z_{2,1} + z_{2,2}) & w_2 &= (0, 0)\\
f_3(z_{3,1}, z_{3,2}) &\equiv 0 & w_3 &= (5, 8).
\end{aligned}
$$

The two goods are complementary for player 1. For example, the goods could be left shoes and right shoes. Player 2 has a small fixed unit value for either type of good, no matter how much of each he/she is allocated. Player 3 has a nonzero initial endowment and no value for either good.

(a) Find the core of the induced cooperative game.

Solution: First find the coalitional value function v. A coalition has zero value unless it includes at least one of the first two agents, and agent 3. If it includes at least one of the first two agents and agent 3, then for maximizing value it is always optimal to take $z_3 = 0$. This gives $v(\{1,3\}) = \sqrt{5}$ and $v(\{2,3\}) = 1.3$.

If all three agents cooperate, it must be that $z_{1,1} = z_{1,2}$, otherwise agent 1 would have an excess of one type of good with zero value, while agent 2 always has positive unit value for either good. Thus, the allocation has the form $z_1 = (a, a)$ and $z_2 = (5-a, 8-a)$ where $0 \le a \le 5$, giving total value $a^{1/2} + (0.1)(5 - a + 8 - a)$, which, since $(a^{1/2})'\Big|_{a=5} = \frac{1}{2\sqrt{5}} = 0.2236 > 0.2$, is achieved by $a = 5$. In other words, the final allocation that maximizes the sum of values is $z^* = ((5, 5), (0, 3), (0, 0))$, yielding $v(\{1, 2, 3\}) = \sqrt{5} + 0.3$.

The core is thus the set of all vectors $x \in \mathbb{R}^3$ such that:

$$
\begin{aligned}
x_i &\ge 0 \quad \text{for } i \in \{1, 2, 3\}\\
x_1 + x_2 + x_3 &= \sqrt{5} + 0.3 \approx 2.5361\\
x_1 + x_3 &\ge \sqrt{5} \approx 2.2361\\
x_2 + x_3 &\ge 1.3
\end{aligned}
$$

(b) Find the set of all competitive equilibria (z^*, p^*) and the set of all payoff profiles $x^* \in \mathbb{R}^3$ for competitive equilibria.

Solution: For a price vector p, the optimal response functions for the agent 1 are given by $\arg\max_{z_i} f_i(z_i) - pz_i$. The response of agent 1 to positive prices has the form $z_1 = (a, a)$

where a maximizes $a^{1/2} - a(p_1 + p_2)$. The optimal response functions are given by:

$$
\begin{aligned}
(z_{1,1}, z_{1,2}) &= \frac{1}{4(p_1 + p_2)^2} \\
(z_{2,i}) &= \begin{cases} 0 & p_i < 0.1 \\ \text{arbitrary number in } \mathbb{R}_+ & p_i = 0.1 \\ +\infty & p_i > 0.1 \end{cases} \\
z_3 &\equiv 0
\end{aligned}
$$

As already found in part (a), the social welfare is maximized at $z^* = ((5,5),(0,3),(0,0))$. The corresponding prices must satisfy $p_1^* + p_2^* = \frac{1}{2\sqrt{5}} = 0.2236$ and $p_2^* = 0.1$. Thus, $p^* = \left(\frac{1}{2\sqrt{5}} - 0.1, 0.1\right) \approx (0.1136, 0.1)$ is the unique solution of the dual problem. In summary, (z^*, p^*) is the unique competitive equilibrium. The corresponding payoff profile is $x^* = (\sqrt{5} - 5(p_1^* + p_2^*), 0, 5p_1^* + 8p_2^*) = \left(\frac{\sqrt{5}}{2}, 0, \frac{\sqrt{5}}{2} + 0.3\right) \approx (1.1180, 0, 1.4180)$.

(c) Find the Shapley value profile.

Solution: The payoff profiles for the orders of arrival of agents are given by:

123	$(0, 0, \sqrt{5} + 0.3)$
132	$(0, 0.3, \sqrt{5})$
213	$(0, 0, \sqrt{5} + 0.3)$
231	$(\sqrt{5} - 1, 0, 1.3)$
312	$(\sqrt{5}, 0.3, 0)$
321	$(\sqrt{5} - 1, 1.3, 0)$

Averaging the six payoff profiles gives

$$x^{Shapley} = \left(\frac{\sqrt{5}}{2} - \frac{1}{3}, \frac{19}{60}, \frac{\sqrt{5}}{2} + \frac{19}{60}\right) \approx (0.7847, 0.3167, 1.4347).$$