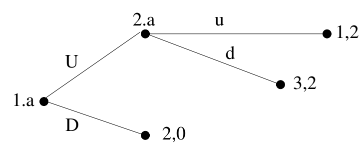
University of Illinois **Fall 2017**

ECE 586BH: Problem Set 4: Problems and Solutions
Extensive form (aka sequential) games

Due: Tuesday, October 31, at beginning of class
Reading: Course notes, Chapter 4

1. **[The subgame perfect equilibria for a perfect information game with a tie]**
Consider the extensive form game with perfect information shown below. For such games, the subgame perfect equilibria are exactly those equilibria that can be produced by the backwards induction algorithm.

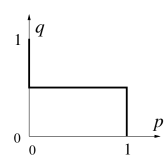


(a) Find the set of all subgame perfect equilibria in pure strategies.

Solution: Let's solve by the backwards induction algorithm. First, player 2 (for set 2.a) can use either u or d because both give a payoff of 2. If player 2 selects u then the payoff vector at 2.a becomes (1,2), so the unique best response of player 1 is D. That is (D, u) is a subgame perfect equilibrium. Similarly, (U, d) is a subgame perfect equilibrium. Those are the only two subgame perfect equilibria in pure strategies.

(b) Find the set of all subgame perfect equilibria in mixed strategies.

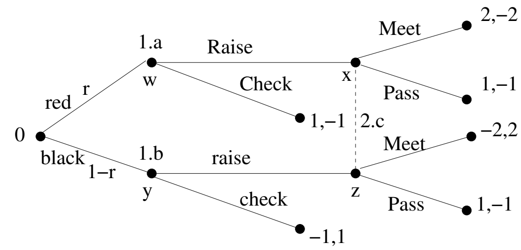
Solution: Let's proceed as in (a), but allow mixed strategies. Player 2 can use $(q, 1-q)$ for any $q \in [0, 1]$, which makes the payoff vector at node 2.a, $(q + 3(1 - q), 2)$, or equivalently, $(3 - 2q, 2)$. Then the strategy of player 1 must be U (i.e. use U with probability $p = 1$) if $0 \leq q < 0.5$, D if $0.5 < q \leq 1$, and $(p, 1 - p)$ for any $p \in [0, 1]$ if $q = 0.5$. The locus of all subgame perfect equilbria is shown in the following figure:



(c) Does this game have a Nash equilibrium (possibly in mixed strategies) that is not subgame perfect?

Solution: No, the equilibria found in part (b) include all Nash equilibria. An explanation for this is that any strategy $(q, 1 - q)$ is optimal for information set 2.a, so any strategy profile is a Nash equilibrium for the subgame rooted at 2.a. Therefore, the sets of Nash equilibria and subgame perfect equilibria coincide.

2. **[Call my bluff card game with bias]**
Fix r with $0 < r < 1$ and consider the following version of the call my bluff card game. The special case $r = 0.5$ is equivalent to the call by bluff card game example in the notes.



(a) Find the expected payoffs for the normal form version of the game.

Solution: Since it is a zero sum game, we list only the payoff of the first player.

	M	P
Rr	$4r-2$	1
Rc	$3r-1$	$2r-1$
Cr	$3r-2$	1
Cc	$2r-1$	$2r-1$

(b) Find a Nash equilibrium of the game in mixed strategies. What is the expected payoff of player 1 at a Nash equilibrium? Your answers should depend on r.

Solution: First we look for pure strategy NE and find that if $r \geq 3/4$ then (Rr, P) is a Nash equilibrium. Player 1 always raises and player 2 always passes, and the payoff to player 1 is 1. If $0 < r < 3/4$ there is no pure Nash equilibrium. To see it, note that Cr and Cc are weakly dominated by Rr and Rc, respectively, so if there existed a pure Nash equilibrium, there would exist a pure Nash equilibrium with player 1 using strategy Rr or Rc. But we have the cycle of unique best responses $M \to Rc \to P \to Rr \to M$ so there can be no pure Nash equilibrium.

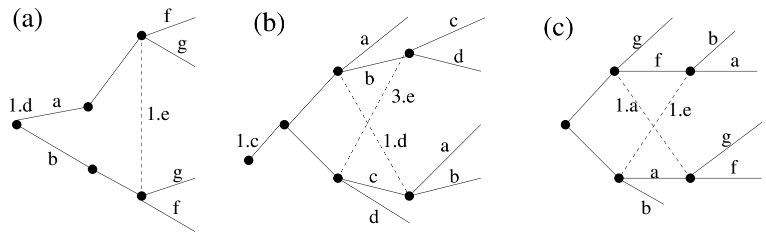
So suppose $0 < r < 3/4$ and seek to identify the Nash equilibria in mixed strategies. Since Cr and Cc are weakly dominated, their can be no weight on them for a Nash equilibrium, so we can restrict attention to strategies Rr and Rc for player 1. Let p denote the probability player 1 uses Rr and q denote the probability player 2 uses M. Then by the equalizer principle, (p, q) represent a Nash equilibrium if and only if

$$p(4r-2) + (1-p)(3r-1) = p + (1-p)(2r-1)$$
$$q(4r-2) + 1 - q = q(3r-1) + (1-q)(2r-1)$$

Or $p = \frac{r}{3(1-r)}$ and $q = \frac{2}{3}$. That is, there is a unique Nash equilibrium in mixed strategies. Player 1 always raises if the card is red and raises with probability $p = \frac{r}{3(1-r)}$ if the card is black. If player 1 raises, player 2 meets with probability $2/3$. Indeed, if player 2 meets with probability $2/3$, the payoff to player 1 is $\frac{8r}{3} - 1$ for either Rr or Rc. Thus, the expected payoff to player 1 at the Nash equilibrium is also $\frac{8r}{3} - 1$.

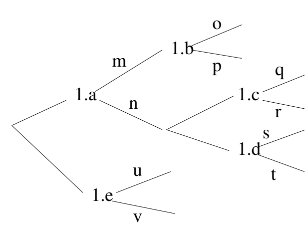
For $r = 3/4$ the Nash equilibria are all strategy pairs of the form $(M, (q, 1-q))$, with expected payoff 1 to player 1.

3. **[Recognizing the total recall property]**
Figures (a)-(c) each show a portion of a sequential game tree with imperfect information. Solid lines indicate directed edges within the game tree, and dashed lines indicate information sets. Which of the figure(s) are consistent with the total recall property? Explain why.



Solution: The graph in (a) violates the total recall property because one state in $1.e$ is reached from $1.d$ using action a at $1.d$ and another state in $1.e$ is reached from $1.d$ using action b. If player 1 knows the state is in information set $1.e$, the player knows that information set $1.d$ was reached earlier, but the player can't recall what action the player took in information set $1.d$.
The graph in (b) is consistent with perfect recall. Player 1 does not need to recall past actions of player 3 and vice versa.
The graph in (c) violates the total recall property. If player 1 is called upon to select an action for information set 1.e, the player cannot accurately recall whether he/she earlier made a decision for information set $1.a$.

4. **[Illustration of Kuhn's theorem]**
Suppose the ordering of the information sets and action labels of player 1 in an extensive form game with perfect recall have the tree ordering pictured:



The edges indicate the possibility of reaching from one information set to the next, with probabilities depending on the probabilities used by nature at randomized nodes, and the strategies of the other players. An example of a pure strategy of player 1 (for the normal form version of the game) is $[moqsu]$, meaning that action m is to be taken at information set $1.a$, action o is to be taken at information set $1.b$, and so on.

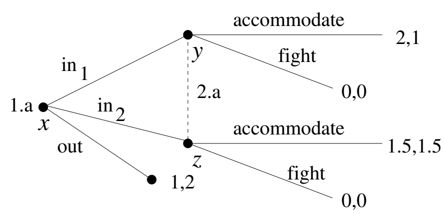
(a) Let τ_1 be the mixed strategy for player 1 for the normal form version of the game given by: $\tau_1 = (0.5)[moqsu] + (0.2)[mprtv] + (0.2)[noqsu] + (0.1)[npqtv]$. Find a behavioral strategy $\sigma_1 = (\sigma_{1,s} : s \text{ is an information set of } 1)$, that is behaviorally equivalent to τ_1. Is σ_1 uniquely determined?

Solution: Suppose the entries of σ_1 are listed in alphabetical order of the information sets. Since each information set has two possible actions and their probabilities must sum to one, it is sufficient to give the probability of the first action. With these conventions, σ_1 is given by $\sigma_1 = \left(\frac{7}{10}, \frac{5}{7}, 1, \frac{2}{3}, \frac{6}{10}\right)$. For example, the second entry is $\frac{5}{7}$, meaning that $\sigma_{1,b.1} = \left(\frac{5}{7}, \frac{2}{7}\right)$. That is true because if nature and the other players were to guide the game towards state $b.1$, then the sum of probabilities of the paths reaching $b.1$ is 0.7, and the sum of the probabilities of the paths exiting $b.1$ with the first action shown, namely o, is 0.5, and $\frac{0.5}{0.7} = \frac{5}{7}$.

(b) Repeat part (a) for the choice $\tau_1 = (0.6)[moqsu] + (0.4)[mprtv]$.

Solution: With the notational convention of part (a), a behavioral strategy σ_1 is behaviorally equivalent to τ_1 if and only if $\sigma_1 = (1, 0.6, *, *, 0.6)$ where each $*$ can be filled in by any number from $[0, 1]$. Thus, σ_1 is not uniquely determined, because the probability of reaching information set $1.c$ or $1.d$ under τ_1 is zero, no matter what probabilities nature uses and what strategies other players use.

5. [**Sequential equilibrium for a variation of the entry deterrence game**]
Consider the following variation of the entry deterrence game, which is similar to the variation in Example 4.13 of the notes.



Letters x, y and z in the figure are the names of the three nonleaf nodes, so, for example, the information set $2.a$ is equal to $\{y, z\}$.

(a) Identify the subgames of the game (including the game itself) and identify both of the subgame perfect equilibria in pure strategies. Also, describe the subgame perfect equilibria such that at least one player uses a nondegenerate mixed strategy. (Hint: The subtrees rooted at y or z are not considered to be subgames because player 2, in controlling information set $2.a$, would not necessarily know which state of $2.a$ the game is in.)

Solution: The only subgame is the game itself, rooted at x. Thus, a strategy profile is subgame perfect if and only if it is a Nash equilibrium. The subgame perfect equilibria in pure strategies are $(out, fight)$ and $(in_1, accommodate)$. The subgame perfect strategy profiles such that at least one player uses a nondegenerate mixed strategy are those of the form $(out, (q, 1-q))$ with $0 < q \leq 0.5$. For these strategy profiles, player 2 accommodates with probability q that is small enough that player 1 does not have incentive to enter the market.

(b) Find the set of all sequential equilibria for this game. Justify your answer. (Hint: By definition, sequential equilibria are assessments that are sequentially rational and consistent.)

Solution: There is only one sequential equilibrium, namely, the assessment (σ, μ) such that

$$\sigma = (\sigma_1, \sigma_2) = (in_1, accomodate) \tag{1}$$

$$\mu = (\mu(1.a), \mu(2.a)) = (\mu(x|1.a), (\mu(y|2.a), \mu(z|2.a))) = ((1), (1, 0)). \tag{2}$$

(Note: $\mu(x|1.a) = 1$ has to be the case for any belief vector μ because x is the only state in information set $1.a$. The choice $\mu(y|2.a) = 1$ means player 2 believes the game state is y given the game state is in $2.a$.)

The above assessment is sequentially rational:
(player 1) $u_1(in_1, \sigma_2|i.a, \mu(i.a)) = 2$, which is greater than either $u_1(in_2, \sigma_2|i.a, \mu(i.a)) =$

1.5 or $u_1(fight, \sigma_2|i.a, \mu(i.a)) = 1$.
(player 2) $u_2(accommodate, \sigma_1|2.b, \mu(2.b)) = 1$, which is greater than $u_2(fight, \sigma_1|2.b, \mu(2.b)) = 0$.

The above assessment is consistent:
Let σ^k be any sequence of strategy profiles with completely mixed strategies such that $\sigma = \lim_{k\to\infty} \sigma^k$ and then define μ^k by $\mu^k(y|2.a) = \frac{\sigma_1^k(in_1)}{\sigma_1^k(in_1)+\sigma_1^k(in_2)}$, (and also, $\mu^k(z|2.a) = 1 - \mu^k(y|2.a)$ and $\mu_k(x|1.a) = 1$). Since $\sigma_1^k(in_1) \to 1$ we see $(\sigma, \mu) = \lim_{k\to\infty}(\sigma^k, \mu^k)$.

There are no other sequential equilibria.
Let (σ, μ) be a sequential equilibrium. For information state $2.a$, *accommodate* (strictly) dominates $fight$, so it must be that $\sigma_2 = accommodate$. Since $\sigma_2 = accommodate$, sequential rationality requires $\sigma_1 = in_1$. Thus, (1) holds. For any sequence of completely mixed strategies $(\sigma^k)_{k\geq 1}$ such that $\sigma^k \to \sigma$ it must be that $\sigma_1^k(in) \to 1$, so consistency requires that μ be given by (2).

(c) Give the normal form representation of the game and identify: the Nash equilibrium in pure strategies, the Nash equilibrium in mixed strategies, and the trembling hand perfect equilibrium (in pure or mixed strategies).

Solution: The payoffs for the normal form of the game are given by:

	$accommodate$	$fight$
in_1	$2, 1$	$0, 0$
in_2	$1.5, 1.5$	$0, 0$
out	$1, 2$	$1, 2$

The Nash equilibria in pure strategies are $(in_1, accommodate)$ and $(out, fight)$. Additional Nash equilibria in mixed strategies are given by $(out, (q, 1-q))$ where $0 \leq q \leq 0.5$. In the search for trembling hand equilibria we can focus attention to these Nash equilibria. The first one, namely, $(in_1, accommodate)$ is trembling hand perfect. The others are not, because the strategy of player 2 is weakly dominated by the strategy *acommodate*. (We appeal to the fact that for two player games with use of mixed strategies, a Nash equilibrium is trembling hand perfect if and only if the strategy of each player is not weakly dominated by any other strategy.)