University of Illinois **Fall 2017**

ECE 586GT: Problem Set 1: Problems and Solutions
Analysis of static games

Due: Tuesday, Sept. 12, at beginning of class
Reading: Course notes part 1 (recommended: Menache and Ozdaglar, Part I)

1. **[Guessing 2/3 of the average]**
Consider the following game for n players. Each of the players selects a number from the set $\{1,\ldots,100\}$, and a cash prize is split evenly among the players who's numbers are closest to two-thirds the average of the n numbers chosen.

   (a) Show that the problem is solvable by iterated elimination of *weakly* dominated strategies, meaning the method can be used to eliminate all but one strategy for each player, which necessarily gives a Nash equilibrium. (A strategy μ_i of a player i is called weakly dominated if there is another strategy μ_i' that always does at least as well as μ_i, and is strictly better than μ_i for some vector of strategies of the other players.)

   Solution: Any choice of number in the interval $\{68,\ldots,100\}$ is weakly dominated, because replacing a choice in that interval by the choice 67 (here 67 is (2/3)100 rounded to the nearest integer) would not cause a winning player to lose, while, for some choices of the other players, it could cause a losing player to win. Thus, after one step of elimination, we assume all players select numbers in the interval $\{1,\ldots,67\}$. After two steps of elimination we assume players select numbers in the set $\{1,\cdots,45\}$. After three steps, $\{1,\cdots,30\}$, and so on. At each step the set of remaining strategies has the form $\{1,\ldots,k\}$, and as long as $k\geq 2$ the set shrinks at the next step. So the procedure terminates when all players choose the number one.

   (b) Give an example of a two player game, with two possible actions for each player, such that iterated elimination of weakly dominated strategies can eliminate a Nash equilibrium. (Hint: The eliminated Nash equilibrium might not be very good for either player.)

   Solution: A bimatrix game with $A_1 = A_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ gives such an example. Playing 2 is weakly dominated for each player, and eliminating those choices leads to the Nash equilibrium $(1,1)$. However, $(2,2)$ is also a Nash equilibrium.

   (c) Show that the Nash equilibrium found in part (a) is the unique mixed strategy Nash equilibrium (as usual we consider pure strategies to be special cases of mixed strategies). (Hint: Let k^* be the largest integer such that there exists at least one player choosing k^* with strictly positive probability. Show that $k^*=1$.)

   Solution: Consider a Nash equilibrium of mixed strategies. Let k^* be the largest integer such that there exists at least one player i choosing k^* with strictly positive probability. To complete the proof, we show that $k^*=1$, meaning all players always choose the number one. For the sake of argument by contradiction, suppose $k^*\geq 2$. Let player i denote a player that plays k^* with positive probability. For any choice of strategies of other players, player i has a pure strategy with a strictly positive probability of winning. Since k^* must be a best response for player i, it must therefore also have a strictly positive probability of winning. It is impossible for player i to win if no other chosen numbers are equal to k^*. (Indeed, if player i were the only one to choose k^*, the second highest chosen number would be strictly closer to 2/3 of the average than k^*.) Thus, at least one of the other players must have a strictly positive probability of choosing k^*. But this means that player i^* could strictly increase her payoff by selecting k^*-1 instead of k^* (indeed, such change would never change her from winning to losing, and in case she wins, she would win strictly more with positive probability) which contradicts the requirement that k^* be a best response for player i. This completes the argument by contradiction.

2. **[Equilbria of the volunteer's dilema game]**
Consider the n player normal form game with $n \geq 2$ such that each player has two actions: C (cooperate) or D (defect). A strategy profile is denoted by $s \in \{C, D\}^n$. Suppose the payoff of any player i is given by $u_i(s) = \mathbf{1}_{\{s_i=D\}} - (11)\mathbf{1}_{\{s=(D,\ldots,D)\}}$. Thus, each player gets one unit of payoff for defecting, but gets total payoff -10 if all players play D.

   (a) Identify all pure strategy Nash equilibria.

   Solution: There are n pure Nash strategy profiles, $s^i, 1 \leq i \leq n$, where s^i is the strategy with $s^i_i = C$ and $s^i_j = D$ for all $j \neq i$. For example, $s^3 = (D, D, C, D, D, \ldots, D)$.

   (b) Identify all mixed strategy Nash equilibria.

   Solution: Denote a mixed strategy profile by $p = (p_1, \ldots, p_n)$, where p_i is the probability player i plays C. Suppose p is a Nash equilibrium profile. If $p_i = 1$ for some i, then $p_j = 0$ for all other j, in which case p is equivalent to one of the Nash equilibria in pure strategies found in part (a). So suppose $p_i < 1$ for all i, implying that D is a best response for any player. And C is a best response for players with $p_i > 0$. The possible payoffs for pure strategies by player i are $u_i(C, p_{-i}) = 0$ and $u_i(D, p_{-i}) = 1 - (11)\beta_i$, where β_i is the probability none of the other players cooperate: $\beta_i = \prod_{j:j\neq i}(1-p_j)$. Given our restriction to $p_i < 1$ for all i, Nash equilibrium is equivalent to $1 - (11)\beta_i \geq 0$ with equality if $p_i > 0$. Since $\beta_i = \frac{B}{1-p_i}$ where $B = \prod_{i=1}^n (1-p_i)$, Nash equilibrium becomes:

$$1 - p_i \geq (11)B \quad \text{with equality if } p_i > 0$$

   Suppose $p_i > 0$ for k players. The probabilities for those players must all equal p, where $(1-p) = (11)(1-p)^k$ or $p = 1 - (1/11)^{1/(k-1)}$, which is valid for any k with $2 \leq k \leq n$. Thus, the mixed strategy Nash equilibria, consists of the pure Nash equilibria found in part (a), and mixed strategy profiles p such that, for some k with $2 \leq k \leq n$, k of the p_i's are equal to $1 - (1/11)^{1/(k-1)}$ and the other $n-k$ p_i's are zero.

   (c) Identify the polytope of all correlated equilibria by giving the set of inequalities they satisfy, and find the correlated equilibria with largest sum of payoffs.

   Solution: By definition, a probability distribution p over $\{C, D\}^n$ is a correlated equilibrium for this game if for each player i,

$$\sum_{s_{-i}} p(D, s_{-i}) u_i(D, s_{-i}) \geq \sum_{s_{-i}} p(D, s_{-i}) u_i(C, s_{-i})$$
$$\sum_{s_{-i}} p(C, s_{-i}) u_i(C, s_{-i}) \geq \sum_{s_{-i}} p(C, s_{-i}) u_i(D, s_{-i}),$$

   which, using the definition of the payoff function u_i, translates to the following constraints, which must be satisfied for all i:

$$\left(\sum_{s_{-i}} p(D, s_{-i})\right) - (11)p((D, \ldots, D)) \geq 0$$

$$0 \geq \left(\sum_{s_{-i}} p(C, s_{-i})\right) - (11)p(s^i),$$

where s^i represents the strategy profile with a single C in the i^{th} position, as defined in part (a).

The sum of payoffs for any pure strategy profile s is equal to $n - 11$ if $s = (D, \ldots, D)$ and otherwise it is equal to the number of C's in s. So the maximum sum of payoffs for any pure strategy profile is $n - 1$, and is achieved if and only if s is one of the n strategy profiles s_i, $1 \leq i \leq n$, found in part (a). Hence, the expected sum of payoffs for any probability distribution p over the set of all strategies is greater than or equal to $n - 1$, with equality if and only if the probability distribution assigns probability one to the set $\{s^1, \ldots, s^n\}$. Since the strategy profiles s_i, $1 \leq i \leq n$, are Nash equilibria, any probability distribution that assigns probability one to the set $\{s^1, \ldots, s^n\}$ is a correlated equilibrium, and such equilibria are the correlated equilibria that maximize the expected payoff.

3. **[Provisioning a public good]**
Suppose n players are invited to contribute payments for a pubic good, such as pavement for a road, a well for water, or a fireworks display, that will be valued by all players. Each player i decides an amount p_i to pay. A strategy profile is denoted by $p = (p_1, \ldots, p_n)$, and the total sum of payments is denoted by $P = p_1 + \cdots + p_n$. Suppose the total sum is used to invest in a public good that is worth $a \ln(1 + P)$ to all players for some fixed and known $a > 0$, so the payoff function of each player i is $u_i(p) = a \ln(1 + P) - p_i$.

   (a) Identify all the Nash equilibrium points for this n-player game (in pure strategies). Also, find the value of the total welfare, $\sum_{i=1}^{n} u_i(p)$, at Nash equilibrium. (Hint: The form of your answer may be different for different values of a.)

   Solution: $u_i(p)$ is a strictly concave function of p_i for p_{-i} fixed, and

$$\frac{\partial u_i(p)}{\partial p_i} = \frac{a}{1 + P} - 1 = \frac{a}{1 + P_{-i} + p_i} - 1,$$

   where P_{-i} is the sum of payments of the other players. The unique best response of player i to the other players is $B(p_{-i}) = (a - 1 - P_{-i})_+$. If $a \leq 1$ then $B_i(p_{-i}) \equiv 0$ for all i; the best response of any player is to pay zero, and the unique Nash equilbrium is $(0, \ldots, 0)$ and all payoffs are 0. If $a > 1$ then the best response of a player i is to pay enough that the total P satisfies $P = a - 1$ if $P_{-i} < a - 1$, and $p_i = 0$ if $P_{-i} \geq a - 1$. Therefore, any payment profile p such that $\sum_{i=1}^{n} p_i = a - 1$ is a Nash equilibrium. (It would seem fair that players pay equal amounts so maybe they could agree to settle on a Nash equilibrium with equal payments, $\frac{a-1}{n}$ each. Still, $(a - 1, 0, \ldots, 0)$ is a Nash equilibrium payment profile, and players 2 through n might be considered "free riders," taking advantage of player 1 paying for the public good alone.)

   The social welfare at Nash equilibrium is $na \ln a - (a - 1)$.

(b) Identify the maximum possible social welfare, which is the maximimum over p of $\sum_i u_i(p)$. Compare to the welfare found in part (a) for $a = 2$ and large n.

Solution: The social welfare is $na\ln(1+P) - P$, which is maximized by $P = 0$ if $a \leq \frac{1}{n}$ and by $P = na - 1$ if $na > 1$. (it doesn't matter which players pay but again they might agree to pay equally) and the maximum social welfare is $na\ln(na) - na + 1$.

For $a = 2$ the welfare at Nash equilibrium is $n(2\ln 2) - 1$ whereas the maximum social welfare is $n(2\ln(2n) - 2) + 1$. Or linear in n vs. $n\ln n$ growth with n. Under the Nash equilibrium the total investment does not grow with n. It suggests that if the players could enter into a binding agreement with each other to pay more, they could all have larger payoffs than under the Nash equilibrium.

4. **[Bertrand equilibrium]**

Suppose n players, for some $n \geq 2$, represent firms that can each produce a common good at a cost c per unit of good. Suppose the action of each player is to declare a price p_i per unit of good. Suppose there is an aggregate demand of consumers such that if the lowest price offered by any firm is p_{min} then the consumers purchase a total quantity $(a - p_{\min})_+$ of goods, where a is a constant with $a > c$, and they purchase an equal amount from each player offering the minimum price. The game is among the players offering prices; the consumers are not considered to be part of the game.

(a) Find the set of all Nash equilibrium profiles $(p_1, \ldots, p_n)$. The form of your answer may depend on the values of a and c

Solution: Consider a strategy profile $p = (p_1, \ldots, p_n)$ with minimum price $p_{\min}$. If $p_{\min} < c$, some player has a negative payoff and could increase payoff to zero by switching to a large price, so p can't be a Nash equilibrium if $p_{\min} < c$. If $p_{\min} \geq a$, the demand is zero, so no player sells any good, and all payoffs are zero. A player could get a larger payoff by decreasing his/her price to something in the interval (c, a), so p can't be a Nash equilibrium if $p_{\min} \geq a$. If $c < p_{\min} < a$, one or more players has positive payoff. A player with zero payoff could get a positive payoff if he/she changed price to $p_{\min}$, so if p were a Nash equilibrium it must be that $p_i = p_{\min}$ for all i and the payoff of every player would be $(p_{min} - c)(a - p_{min})/n > 0$. But that is not a Nash equilibrium because if one of the players slightly dropped his/her price by a small $\epsilon > 0$, then the player would serve the entire demand and get payoff $(p_{\min} - \epsilon - c)(a - p_{\min} + \epsilon)$ which is larger than before for ϵ small enough, because $n \geq 2$. So p is not a Nash equilibrium if $c < p_{\min} < a$. There is only one possibility of $p_{\min}$ left: a necessary condition for p to be a Nash equilibrium is $p_{\min} = c$.

If $p_{\min} = c$ then all players have payoff of zero. However, if exactly one player i has $p_i = c$, then that player could increase his/her payoff to a strictly positive amount by increasing p_i by a small amount, so that $p_i > c$ and p_i is still the unique minimum price among all players. If $p_{\min} = c$ and at least two players have price $p_i = c$, then all players receive zero payoff, and no player can achieve a strictly positive payoff by unilaterally changing price. In summary, p is a Nash equilibrium if and only if $p_{\min} = c$ and $p_i = c$ for at least two players. Basically, the competition among the players drives the sum of payoffs to zero at Nash equilibrium as soon as $n \geq 2$. In contrast, for the Cournot competition, the sum of payoffs converges to zero as $n \to \infty$, but it is positive for any finite n. This effect is known as the Bertrand paradox.

(b) Suppose the production costs vary by player, with the per unit production cost of player i given by some $c_i > 0$. For simplicity, suppose $c_1 < c_2 < \cdots < c_n$ and suppose $c_1 < a$.

Find the set of all Nash equilbrium profiles $(p_1, \ldots, p_n)$.

Solution: If player 1 were the only player, than the payoff of player 1 would be $(p_1 - c_1)(a - p_1)_+$, which is negative for $p_1 < c_1$, strictly increasing for $p_1 \leq \frac{a+c_1}{2}$, strictly decreasing for $\frac{a+c_1}{2} \leq p_1 \leq a$ and zero for $p_1 \geq a$. The payoff is maximized at $p_1 = \frac{a+c_1}{2}$. Consider now the game for $n \geq 2$, and consider two cases:

Case 1: Suppose $\frac{a+c_1}{2} \leq c_2$. If player 1 ignores the other players and sets $p_1 = \frac{a+c_1}{2}$ as in the monopoly situation, then none of the other players could have a positive payoff. For case 1, a price profile (p_i) is a Nash equilbrium if and only if $p_1 = \frac{a+c_1}{2}$ and $p_i > \frac{a+c_1}{2}$ for $i \neq 1$.

Case 2: Suppose $\frac{a+c_1}{2} > c_2$. We show there is no Nash equilibrium in this case. Let $p_{\min} = \min_i p_i$.

Subcase 2.1: If $p_{\min} < c_2$ then it must be that $c_1 < p_1 = p_{\min} < p_j$ for $j \geq 2$ because if any player other than player 1 sold a positive quantity of good they would have a negative payoff, and player 1 can achieve a positive payoff with a price greater than c_1. However, the payoff of player 1 is strictly increasing in p_1 in the monopoly situation for $c_1 \leq p_1 \leq \frac{a+c}{2}$, and hence for $c_1 \leq p_1 \leq c_2$ in this subcase. So it wouldn't be a Nash equilbrium if $p_1 < c_2$. Basically, player 1 could get a small increase in payoff by increasing p_1 a small amount. So there are no Nash equilbria in Subcase 2.1.

Subcase 2.2: If $p_{\min} > c_2$ then players 1 and 2 could each get a strictly positive payoff by changing their price, if their payoff was zero. So at a Nash equilibrium they must have $p_1 = p_2 = p_{\min}$. But then either of the two players could strictly increase their payoff by slightly decreasing their price. So there are no Nash equilbria in Subcase 2.2.

Subcase 2.3: If $p_{\min} = c_2$ at a Nash equilibrium, it must be that $p_1 = c_2$, because otherwise player 1 could switch from zero payoff to strictly positive payoff by changing p_1 to c_2. So we have $p_{\min} = p_1 = c_2 < a$. If $p_2 = c_2$ as well, player 1 is sharing the demand with player 2, so player 1 could increase his/her profit by slightly decreasing p_1. If $p_2 > c_2$, and since $\frac{a+c_1}{2} > c_2$, player 1 could increase his/her payoff by slightly increasing p_1 while still capturing the total load. So there are no Nash equilbria in Subcase 2.3.

So there are no Nash equilibria in Case 2.

5. **[Nash saddle point]**
Consider a two person zero sum game represented by a finite $m \times n$ matrix A. Player 1 selects a probability vector p and player 2 selects a probability vector q. Player 1 wishes to minimize pAq^T and player 2 wishes to maximize pAq^T. Let V denote the value of the game, so $V = \min_p \max_q pAq^T = \max_q \min_p pAq^T$.

   (a) Consider the following statement S: If $\overline{p}$ and $\overline{q}$ are probability distributions (of the appropriate dimensions) such that $\overline{p}A\overline{q}^T = V$, then $(\overline{p}, \overline{q})$ is a Nash equilibrium point (in mixed strategies). Either prove that statement S is true, or give a counter example.

   Solution: The statement if false. For example, let $m = n = 2$ and $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $\overline{p} = (1, 0)$ and $\overline{q} = (0, 1)$. Then $\overline{p}A\overline{q}^T = 0 = V$. But if the second player were to switch from using $\overline{q}$ to using $\widehat{q} = (1, 0)$, the payoff would increase from zero to $\overline{p}A\widehat{g}^T = 1$. Therefore, $(\overline{p}, \overline{q})$ is not a Nash equilibrium.

   (b) Consider the following statement T: A Nash equilibrium consisting of a pair of pure strategies exits if and only if $\min_i \max_j A_{i,j} = \max_j \min_i A_{i,j}$. Either prove that statement T is true, or give a counter example.

Solution: The statement is true. It is the same as a statement proved in class and the notes, but with p replaced by i and q replaced by j. The proof is repeated here.

(if part) Suppose (i^*, j^*) is a Nash equilibrium. By definition, this means $\min_i A_{i,j^*} = A_{i^*,j^*} = \max_j A_{i^*,j}$. It is easy to see that $\min_i \max_j A_{i,j}$ and $\max_j \min_i A_{i,j}$ are both contained in the length zero interval $[\min_i A_{i,j^*}, \max_j A_{i^*,j}]$, so they must be equal.

(only if part) Suppose $\min_i \max_j A_{i,j} = \max_j \min_i A_{i,j}$. Let i^* be a minmax optimal action for the first player and j^* be a maxmin optimal action for the second player. Then $\max_j A_{i^*,j} = \min_i \max_j A_{i,j} = \max_j \min_i A_{i,j} = \min_i A_{i,j^*}$. Therefore, $A_{i^*,j^*} \leq \max_j A_{i^*,j} = \min_i A_{i,j^*}$ and $A_{i^*,j^*} \geq \min_i A_{i,j^*} = \max_j A_{i^*,j}$, which by definition means (i^*, j^*) is a Nash equilibrium.