University of Illinois **Fall 2017**

ECE 586BH: Problem Set 2

Existence and uniqueness of Nash equilibria, evolutionary dynamics

Due: Tuesday, Sept. 26, at beginning of class
Reading: Course notes, Chapter 1 (sections on existence and uniqueness) and Chapter 2

1. **[Max flow min cut theorem as an example of linear programming duality]**
Let (V, E, C, s, t) be a directed graph with vertex set V, edge set E, nonnegative edge capacities $C = (C_e : e \in E)$, and distinct vertices $s, t \in V$ representing the source and terminus of flow. Extend $C_{u,v}$ to all of $V \times V$ by letting $C_{u,v} = 0$ if $(u, v) \notin E$. The max flow problem can be written as:

$$\max_f \sum_{u \in V \backslash t} (f(u,t) - f(t,u))$$

$$\text{subject to:} \quad f_{u,v} \in [0, C_{u,v}] \qquad \text{for } u, v \in V \times V$$

$$\sum_{u \in V \backslash v} f(u,v) - \sum_{w \in V \backslash v} f(v,w) = 0 \qquad \text{for } v \in V \backslash \{s,t\}.$$

The second set of constraints represents conservation of flow.

   (a) Show the dual linear programming problem can be expressed as:

$$\min_\lambda \sum_{(u,v) \in V \times V} C_{u,v}(\lambda_v - \lambda_u)_+$$

$$\text{subject to: } \lambda_s = 0, \lambda_t = 1.$$

   Hint: There is no need to introduce multipliers for the constraints $f_{u,v} \in [0, C_{u,v}]$.

   (b) Show the dual problem is equivalent to the min cut problem. (By definition, an $s - t$ cut is a partition $(S, V \backslash S)$ of V such that $s \in S$ and $t \in V \backslash S$. The value of such a cut is $\mathsf{value}(S) = \sum_{(u,v) \in S \times (V \backslash S)} C(u, v)$. The min cut problem is to find an $s - t$ cut with minimum value.) (Hint: Show it can be assumed without loss of generality that $\lambda_v \in [0, 1]$ for all v. Then, $\phi(\lambda) = \int_0^1 \mathsf{value}(S(\tau)) d\tau$, where $S(\tau) = \{u : \lambda_u < \tau\}$.)

2. **[Existence and uniqueness of NE for two games with continuous type strategies]**
This problem concerns an n player game with strategy space $S_i = [0, 1]$ for all players i, and space of strategy vectors $S = S_1 \times \cdots \times S_n$.

   (a) Given a vector of strategies $x \in S$, let $\overline{x} = \frac{x_1 + \cdots + x_n}{n}$. Consider the payoff functions $u_i(x) = c x_i(1 - x_i) - \frac{1}{2}(x_i - \overline{x})^2$, where $c \geq 0$. For what values of $c \geq 0$ does there exist a pure strategy Nash equilibrium?

   (b) For the payoff functions of part (a), for what values of c is the pure strategy Nash equilibrium unique?

   (c) Now consider the payoff functions $u_i(x) = \frac{c}{2} x_i(1 - x_i) + x_i \left(\sum_{j=1}^n a_{i,j} x_j^3 \right)$, where $|a_{i,j}| \leq 1$ and $a_{i,i} = 0$ for all i, j. Find a constant c_0 so that for $c \geq c_0$ there exists a pure strategy Nash equilibrium.

(d) For the payoff functions of part (c), give a value c_1 so that there is a unique pure strategy NE if $c > c_1$. (Hint: A sufficient condition for a symmetric matrix to be negative definite is that the diagonal elements be strictly negative, and the sum of the absolute values of the off-diagonal elements in any row be strictly smaller than the absolute value of the diagonal element in the row.)

3. **[Evolutionarily stable strategies and states]**

Consider the following symmetric, two-player game:

	1	2
1	0,0	1,2
2	2,1	0,0

That is, each player selects 1 or 2. If they select different numbers, the payoffs are the numbers selected. If they select the same number, the payoffs are zero.

(a) Does either player have a (weaky or strongly) dominant strategy?

(b) Identify all the pure strategy and mixed strategy Nash equilibria.

(c) Identify all evolutionarily stable pure strategies and all evolutionarily stable mixed strategies.

(d) The replicator dynamics based on this game represents a large population consisting of type 1 and type 2 individuals. Show that the evolution of the population share vector $\theta(t)$ under the replicator dynamics for this model reduces to a one dimensional ordinary differential equation for $\theta_t(1)$, the fraction of the population that is type 1.

(e) Identify the steady states of the replicator dynamics.

(f) Of the steady states identified in the previous part, which are asymptotically stable states of the replicator dynamics? Justify your answer.

4. **[Evolutionarily stable strategies and states, II]**

Consider the following symmetric, two-player game:

	1	2	3
1	0,0	1,2	1,3
2	2,1	0,0	2,3
3	3,1	3,2	0,0

That is, each player selects 1,2, or 3. If they select different numbers, the payoffs are the numbers selected. If they select the same number, the payoffs are zero.

(a) Identify all the pure strategy and mixed strategy Nash equilibria.

(b) Identify all evolutionarily stable pure strategies and all evolutionarily stable mixed strategies.

(c) Identify the steady states of the replicator dynamics.

(d) Prove that the strategy (or one of the strategies) identified in the previous part, when used by both players in a two player game, is a trembling hand perfect equilibrium. (Use a proof based directly on the definition of trembling hand perfect equilibrium.)

5. **[Simulation of evolutionary game of doves and hawks]**

The dove hawk game is the two player symmetric static form game given by:

	D	H
D	4,4	1,5
H	5,1	0,0

(a) Find an evolutionarily stable strategy (ESS) and show that it is unique.

(b) For this part you need to write and run a computer simulation using a random number generator (i.e. Monte Carlo simulation). You are to simulate a population of doves and hawks in discrete time. Suppose there are initially $n_D(1)$ dove's and $n_H(1)$ hawks at the initial time, $t = 1$. Given the numbers of each type at time t, $(n_D(t), n_H(t))$, the numbers at time $t + 1$ are determined as follows. Two distinct birds are selected from among all $n_D(t) + n_H(t)$ birds present at time t, and the two birds play the above two player game (where the strategy of a bird is the type of the bird). After the game, the two birds are returned to the population. In addition, for each player, more birds of the same type as that player are added to the population as well, with the number added equal to the payoff of the player. For example, if both birds are doves, they each have payoff 4, so the two doves are returned, plus a total of eight more doves (because 8=4+4) are added to the population. Turn in (1) a copy of your computer code and (2) a graph showing the number of doves and the number of hawks versus time t for $1 \leq t \leq 100$, beginning with one dove and ten hawks at time $t = 1$. Please take a look at the iPython notebook: http://nbviewer.jupyter.org/urls/courses.engr.illinois.edu/ece586GT/fa2017/ece586GT_ps2.ipynb?flush_cache=true. You are free to make use of the notebook for your assignment, for example by modifying some part of it.