University of Illinois **Fall 2017**

ECE 586BH: Problem Set 1
Analysis of static games

Due: Tuesday, Sept. 12, at beginning of class
Reading: Course notes part 1 (recommended: Menache and Ozdaglar, Part I)

1. [**Guessing 2/3 of the average**]
Consider the following game for n players. Each of the players selects a number from the set $\{1, \ldots, 100\}$, and a cash prize is split evenly among the players who's numbers are closest to two-thirds the average of the n numbers chosen.

   (a) Show that the problem is solvable by iterated elimination of *weakly* dominated strategies, meaning the method can be used to eliminate all but one strategy for each player, which necessarily gives a Nash equilibrium. (A strategy μ_i of a player i is called weakly dominated if there is another strategy μ_i' that always does at least as well as μ_i, and is strictly better than μ_i for some vector of strategies of the other players.)

   (b) Give an example of a two player game, with two possible actions for each player, such that iterated elimination of weakly dominated strategies can eliminate a Nash equilibrium. (Hint: The eliminated Nash equilibrium might not be very good for either player.)

   (c) Show that the Nash equilibrium found in part (a) is the unique mixed strategy Nash equilibrium (as usual we consider pure strategies to be special cases of mixed strategies). (Hint: Let k^* be the largest integer such that there exists at least one player choosing k^* with strictly positive probability. Show that $k^* = 1$.)

2. [**Equilbria of the volunteer's dilema game**]
Consider the n player normal form game with $n \geq 2$ such that each player has two actions: C (cooperate) or D (defect). A strategy profile is denoted by $s \in \{C, D\}^n$. Suppose the payoff of any player i is given by $u_i(s) = \mathbf{1}_{\{s_i = D\}} - (11)\mathbf{1}_{\{s=(D,\ldots,D)\}}$. Thus, each player gets one unit of payoff for defecting, but gets total payoff -10 if all players play D.

   (a) Identify all pure strategy Nash equilibria.

   (b) Identify all mixed strategy Nash equilibria.

   (c) Identify the polytope of all correlated equilibria by giving the set of inequalities they satisfy, and find the correlated equilibria with largest sum of payoffs.

3. [**Provisioning a public good**]
Suppose n players are invited to contribute payments for a pubic good, such as pavement for a road, a well for water, or a fireworks display, that will be valued by all players. Each player i decides an amount p_i to pay. A strategy profile is denoted by $p = (p_1, \ldots, p_n)$, and the total sum of payments is denoted by $P = p_1 + \cdots + p_n$. Suppose the total sum is used to invest in a public good that is worth $a\ln(1 + P)$ to all players for some fixed and known $a > 0$, so the payoff function of each player i is $u_i(p) = a\ln(1 + P) - p_i$.

   (a) Identify all the Nash equilibrium points for this n-player game (in pure strategies). Also, find the value of the total welfare, $\sum_{i=1}^{n} u_i(p)$, at Nash equilibrium. (Hint: The form of your answer may be different for different values of a.)

(b) Identify the maximum possible social welfare, which is the maximimum over p of $\sum_i u_i(p)$. Compare to the welfare found in part (a) for $a = 2$ and large n.

4. **[Bertrand equilibrium]**
Suppose n players, for some $n \geq 2$, represent firms that can each produce a common good at a cost c per unit of good. Suppose the action of each player is to declare a price p_i per unit of good. Suppose there is an aggregate demand of consumers such that if the lowest price offered by any firm is p_{min} then the consumers purchase a total quantity $(a - p_{\min})_+$ of goods, where a is a constant with $a > c$, and they purchase an equal amount from each player offering the minimum price. The game is among the players offering prices; the consumers are not considered to be part of the game.

   (a) Find the set of all Nash equilibrium profiles $(p_1, \ldots, p_n)$. The form of your answer may depend on the values of a and c

   (b) Suppose the production costs vary by player, with the per unit production cost of player i given by some $c_i > 0$. For simplicity, suppose $c_1 < c_2 < \cdots < c_n$ and suppose $c_1 < a$. Find the set of all Nash equilbrium profiles $(p_1, \ldots, p_n)$.

5. **[Nash saddle point]**
Consider a two person zero sum game represented by a finite $m \times n$ matrix A. Player 1 selects a probability vector p and player 2 selects a probability vector q. Player 1 wishes to minimize pAq^T and player 2 wishes to maximize pAq^T. Let V denote the value of the game, so $V = \min_p \max_q pAq^T = \max_q \min_p pAq^T$.

   (a) Consider the following statement S: If $\overline{p}$ and $\overline{q}$ are probability distributions (of the appropriate dimensions) such that $\overline{p}A\overline{q}^T = V$, then $(\overline{p}, \overline{q})$ is a Nash equilibrium point (in mixed strategies). Either prove that statement S is true, or give a counter example.

   (b) Consider the following statement T: A Nash equilibrium consisting of a pair of pure strategies exits if and only if $\min_i \max_j A_{i,j} = \max_j \min_i A_{i,j}$. Either prove that statement T is true, or give a counter example.