(Revealed) Optimal Mechanisms (Myerson '82, presentation following Krishna Chap. 5.)

Seller has a single object to sell

Buyers have independent private values \((x_i)_{i \in I}\)

\[ x_i \in [0, \omega_i], \text{ pdf } f_i, \text{ CDF } F_i \]

A selling mechanism is a triple \((B, \pi, \mu)\)

\(B\) = space of bid vectors

\(\pi : B \rightarrow \Delta^{I}\) = probabilistic allocation rule

\(\mu : B \rightarrow \mathbb{R}^N\) = payment rule

A mechanism induces a game of incomplete information among the buyers, where the type of buyer \(i\) is his value \(x_i\) for the object.

Following Myerson '82, we seek the mechanism such that for a specified Bayesian equilibrium, the expected revenue is maximized.
Revelation Principle

A direct mechanism \((Q, M)\) consists of:

- Space of \((x_1, \ldots, x_N)\), value vectors 
- \(Q : X \rightarrow \Delta\) — random allocation
- \(M : X \rightarrow \mathbb{R}^N\) — payment rule

**Proposition** Given a mechanism \((\beta, \pi, \mu)\) and an equilibrium \(\beta\) for that mechanism, there exists a direct mechanism such that reporting truthfully is an equilibrium, with the same outcomes as \(\mu\) for every \(x_i\) (i.e., some dist\(^n\) of winner and some payoffs.)

**Proof** \(\beta_i(x_i)\) is response of player \(i\) for the equilibrium.

\[
\begin{array}{c}
x \\
\mapsto \\
\beta(\cdot) \\
\mapsto \\
\text{Bid} \\
\mapsto \\
\text{vector} \\
\mapsto \\
\text{winner dist\(^n\)} \\
\mapsto \\
\text{payment vector} \\
\mapsto \\
\mu(\cdot) \\
\mapsto \\
Q(x) = \pi(\beta(x)) \\
M(x) = \mu(\beta(x))
\end{array}
\]
Incentive Compatibility (IC)

If buyer \( i \) bids \( z_i \), she is
allocated the object with probability

\[- q_i(z_i) = \sum_{x_i} q_i(z_i, x_i) f_{x_i}(x_i) \, dx_i \]

and she expects to pay:

\[- m_i(z_i) = \sum_{x_i} m_i(z_i, x_i) f_{x_i}(x_i) \, dx_i \]

Incentive compatibility for buyer \( i \) is
determined entirely by the functions
\( q_i, m_i \) over the range of possible
values \( x_i \in [0, \omega_i] \):

\[ IC: U(x_i) \equiv q_i(x_i) x_i - m_i(x_i) \geq q_i(z_i) x_i - m_i(z_i) \]

(Note: constraints are linear.)
For \( x < y \),

\[
q(x) x - m(x) > q(y) x - m(y)
\]

\[
+ q(y) y - m(y) > q(x) y - m(x)
\]

\[
(g(x) - g(y))(x - y) \geq 0 \quad \text{\( \therefore \) \( g \) nondecreasing}
\]

It implies (recall \( U_i(x_i) = g_i(x_i)x_i - m_i(x_i) \))

\[
U_i(x_i) = \max_{z_i} g_i(z_i) x_i - m_i(x_i) \leq U_i
\]

\( \therefore U_i \) is absolutely continuous (so differentiable) and integral of its derivative.

For \( x_i, z_i \in [0, \omega] \)

\[
U_i(z_i) \geq g_i(x_i) z_i - m_i(x_i) = U_i(x_i) + g_i(x_i)(z_i - x_i)
\]

\( \therefore g_i(x_i) \) is a subgradient of \( U_i \) at \( x_i \).

again see \( g_i \) is nondecreasing.

\( \therefore U_i \) is completely determined by \( U_i(0) \) and \( g_i \)
\[ U(x_i) \]

\[ q_i(x_i) x_i - m_i(x_i) = U_i(0) + \int_0^x q_i(t) \, dt \]

\[ m_i(x_i) = m_i(0) + q_i(x_i) x_i - \int_0^x q_i(t) \, dt \]

\[ \text{Revenue Equivalence} \]

C does not involve \( t \):

So if the direct mechanism \((Q, M)\) is IC then \( g \) is nondecreasing and \( m \) is determined by (*)

Conversely, if \( g \) is nondecreasing and \( m \) is determined by (*) then \((Q, M)\) is IC.

(For converse, follow steps in reverse order)

(*If \( q_i \) is differentiable, \( m_i'(x_i) = x_i g_i'(x_i) \) -- makes sense.)

**Individual Rationality (IR)**

**Dot**: Mechanism is IR if for all \( i, x_i, U_i(x_i) \geq 0 \).

IR = player is not forced to have negative payoff
(a reasonable property for revenue maximization problem -- else seller can just set arbitrarily high payments).

\[ IR \iff U_i(0) \geq 0 \iff m_i(0) \leq 0 \]

\[ g_i(0) \cdot 0 - m_i(0) \]
Illustrating revenue equivalence

Example: Suppose $g_i(x_i) = x_i$ for $0 \leq x_i \leq 1$.

What is the maximum payment function $m_i$ satisfying IR and IC constraints?

$m_i' = g_i' x_i$. Take $m_i(0) = 0$ to maximize revenue.

$m_i' = x_i$

$m_i(x_i) = \frac{1}{2} x_i^2$

Double check: If your true value is $x_i$ and you report $z_i$, your expected payoff is $x_i g_i(z_i) - m_i(z_i)$ or $x_i z_i - \frac{1}{2} z_i^2$.

This is maximized by $z_i = x_i$. ✓

Or $g_i(x_i) = 1 - e^{-x_i}$ $x_i > 0$ $\Rightarrow$ $m_i' = x_i g_i' = x_i e^{-x_i}$

$m_i(x_i) = 1 - (1 + x_i) e^{-x_i}$
If player $i$ were playing a second price auction and $q_i$ were the CDF of the highest bid $W$ of the other players, then for bid $x_i$, player $i$ would get item with probability $q_i(x_i)$, and mean payment would be $m_i(x_i) = q_i(x_i) E[W | W < x_i]$. $W$ has CDF $q_i$.

$$m_i(x_i) = \frac{q_i(x_i)}{q_i'} \left( \frac{\int_0^{x_i} t q_i'(t) \, dt}{q_i(x_i)} \right)$$

Integrate by parts:

$$m_i(x_i) = x_i q_i(x_i) - \frac{\int_0^{x_i} q_i(t) \, dt}{q_i'}$$

Same as formula derived for $m_i$ earlier. As expected, given mean payment is determined by function $q_i$ and $m_i(0)$. 

\[ f(x) \]
Using the revelation principle and the revenue equivalence property of IC mechanisms, we now address (revenue) optimal mechanisms.

Given $(f_i)_{i \in [N]}$ where $f_i$ is a pdf on $[0, w_i]$.

\[
\text{maximize} \quad E\left[ \sum_{i} m_i(X_i) \right]
\]
over $(Q, M)$
such that IC, IR.

Take \( m_1(0) = 0 \) (optimal) so \( m_i(x) = g_i(x) - \int_0^x g_i(t) \, dt \)
\[
E[m(x_i)] = \int_0^{w_i} g_i(x_i) x_i f_i(x_i) \, dx_i
\]

Key Step! \[ \Delta = -\int_0^{w_i} \left( \int_0^{w_i} g_i(t) \, dt \right) f_i(x_i) \, dx_i \]

So \[ E[m(x_i)] = \int_0^{w_i} \psi_i(x_i) g_i(x_i) f_i(x_i) \, dx_i = E[\psi_i(x_i) I_{\{\text{win}\}}] \]

where \( \psi_i(x_i) = x_i - \frac{1 - F_i(x_i)}{f_i(x_i)} \) "win" means gets object.

\( \psi_i(x_i) \) is called the \underline{virtual valuation} for buyer $i$. 
So (revenue) optimal mechanism solves:

$$\max E \left[ \sum_{i=1}^{N} \psi_i(x_i) I_{\{i \text{ wins}\}} \right]$$

over $Q$ subject to

$\gamma_i(x_i)$ nondecreasing for each $i$.

(Recall, $Q$ is winner selection function)

If $\psi_i(x_i)$ is increasing in $x_i$ for each $i$ (i.e. the design problem is regular) then

the optimal assignment rule is

Select winner from $\arg \max_{i} \psi_i(x_i)$ if $\max_{i} \psi_i(x_i) > 0$

Select no winner if $\psi_i(x_i) < 0$ for all $i$

Indeed, this rule maximizes $(**)$ over all assignment rules (ignoring the constraint that $\gamma_i(x_i)$ be nondecreasing) but the constraint is satisfied anyway. The revenue is

$$E \left[ \max \{ \psi_1(x_1), \ldots, \psi_N(x_N), 0 \} \right]$$
Payment rule for optimal auction: Need
\[ m_i(x_i) = q_i(x_i) x_i - \int_0^x q_i(t) \, dt \]
This only specifies average payment, and
is paid whether player i wins or not.
Any choice of \( M_i(x) \) with \( E[M_i(x) \mid x_i = x_i] = m_i(x_i) \)
would be revenue optimal subject to IC, ER
constraints. Think of second price auction

can help us determine good choice for \( M_i \).

Let \( M_i(x_{-i}) = \min \{ z_i : \psi_i(z_i) \geq 0 \text{ and } \psi_i(z_i) \geq \psi_i(x_{-i}) \} \)

= minimum value i would need
to win, given \( x_{-i} \).

Then let \( M_i(x) = y_i(x_{-i}) I_{\{i \text{ wins}\}} \)

We can argue that as in a second price auction,
this is the most buyer can charge subject to IC.

As a double check, we check \( E[M_i(x) \mid x_i = x_i] = m_i(x_i) \)
Check

\[ E[M_i(x) \mid x_i = x_i] = E[y_i(x_i) \mathbb{1}_{y_i(x_i) \leq x_i}] \]

\[
= \int_{0}^{x_i} g_i(x_i) - g_i(t) \, dt \\
= x_i g_i(x_i) - \int_{0}^{x_i} g_i(t) \, dt \\
= m_i(x_i) \checkmark
\]

CDF of \( y_i(x_i) \) is \( g_i(t) \)

Note: The reserve value for buyer \( i \) is given by \( r_i = \inf \{ z_i : y_i(z_i) > 0 \} \).

Buyer \( i \) can't win if bid is less than \( r_i \).

If \( x_j < r_i \) for all buyers, seller doesn't allocate. It is assumed seller has free disposal - i.e., no cost to not allocate to a buyer.
Recovery of 2\textsuperscript{nd} Price Auction

\[
\max_{E} \{ X; i | \text{given winner}\}
\]
over (a,m)
subject to E(S_i)

Clearly the selection rule is to select

\[i^* \in \arg \max_{i} X_i, \quad \text{if } W_i = \max_{j \neq i} X_j\]

\[q_i(x_i) = P\{ W_i < x_i \} = \prod_{j \neq i} F_j(x_j) \quad q_i \text{ is CDF of } W_i\]

\[E[Q_i(x)] = W_i I\{ W_i < x \} \quad (2\text{nd price})\]

Check

\[E[M_i(x) | X_i = x_i] = E[W_i I\{ W_i < x_i \}]\]

\[= x_i q_i(x_i) - \int q_i(t) dt\]

as in revenue optimal case.
Auctions with Interdependent Values
(M. Tsiros + Weber 1983 paper is key, presentation here based on Krishna, Chap. 6)

Focus on Bayes-Nash equilibrium for basic auctions.

First look at
Independent Private Values

\[ x_1, \ldots, x_n \text{ independent, } x_i \sim f_i \]

We've discussed 2\textsuperscript{nd} price auction --
weakly dominant strategy is truthful bidding.

English auction -- seller price increases continuously, starting at zero. Buyers decide when to drop out and see each other drop out. At the time only one buyer remains, that buyer wins item and pays the price at that time.

Does a player in English auction have a
dominant strategy? ... Yes, drop when price equal value. Equivalent to second price auction. Even though a buyer $i$ with value $x_i$ sees some others dropping out at prices below $x_i$, that information causes me to need to deviate.

First price (Symmetric case) let

$$y_i = \max\{x_1, \ldots, x_N\}.$$ Suppose other buyers are increasing strategy $\beta$. Conditional payoff of buyer $i$ given $x_i = x$ and bid $b$ is

$$\Pi_i(b, x) = P[\beta(y_i) < b](x-b)$$

$$= P[y_i < \beta(b)](x-b)$$

$$= G(\beta'(b))(x-b) \quad b = \text{CDF of } Y_i = F^{n-1}$$
\[ \frac{d}{db} \left( \frac{G'(\rho'(b))}{\rho'(b)} (x-b) - G(\rho'(b)) \right) = 0 \]

If player 1 also uses \( \beta \), then \( \beta'(b) = x \).

Yields
\[ G'(x) (x-\beta(x)) - G(x) \beta'(x) = 0 \]

\[ G'(x) x = \left( G(x) \beta(x) \right)' \]
\[ g = G' : \text{density} \]

\[ G(x) \beta(x) = \int_0^x y \, g(y) \, dy \]

\[ \beta(x) = \frac{1}{G(x)} \int_0^x y g(y) \, dy \]

This was derived as necessary condition for symmetric Bayes NE. Can show sufficient (Kriste sect. 2.2)

\[ R(\text{Revenue Ranking}) \quad \text{\( R = R \geq R \)} \]

(\text{symmetric independent private valuations})

(We saw 2\textsuperscript{nd} price is revenue optimal.)
- In many situations buyers don't know their values for the object at auction time. Perhaps have signals $x_i$. One model:

$$U_i = g_i(x_1, \ldots, x_n) \text{ or typically}$$

$$\text{(could be } E[U_i|x_1, \ldots, x_n])$$

Moreover, the signals may be dependent.

- What are Bayes-Nash equilibria like for the three auctions?

- Are second price and English auctions still equivalent. (No -- why?)

- What happens to revenue ranking?

Milgrom and Weber (1983) addressed these questions. Assume symmetry, examine symmetric equilibria, assume a strong version of positive correlation of signals.
Def $X_1, \ldots, X_n$ with joint pdf over $\prod_{i=1}^{N} [0, w_i]$

are affiliated (M&W notation, others say —— )

if $f(xy)f(x)f(y) \geq f(x)f(y)$

(i.e. $\log(f(x,y))$ is supermodular):

$$\log f(xy) + \log f(xv y) \geq \log f(x) + \log f(y)$$

(This terminology of Milgrom and Weber has stuck within economics literature. Known to statisticians as total positivity.

We'll not need/use it, but a consequence is the FK5 inequality: $\text{Cov}(g(X), h(X)) \geq 0$

for any non-decreasing functions $g$ and $h$.

(Follows from Ahlswede-Daykin inequality, which has a simple proof by induction in discrete setting, due to Bollman (1980).

See Ahlswede and Blinovski Lectures on Adv. in Combinatorics, Springer 2008)

Some times called association property.
Some consequences of affiliation

1. (Equivalent to affiliation) For any subset \( I \) of the coordinates, the conditional pdf of \( X-I \) given \( X-I \) is monotone nondecreasing from \( (\mathbb{R}^i, \text{pointwise order}) \) to pdfs for \( X-I \) in likelihood ratio order.

\[ \text{E.g. } I=\{k+1, \ldots, n\}: \quad \frac{f(x_j, \ldots, x_k | x_{k+1}, \ldots, x_n)}{f(x_j, \ldots, x_k | x_{k+1}, \ldots, x_n)} \quad \text{pointwise} \]

is weakly increasing in \((x_j, \ldots, x_k)\) if \((x_{k+1}, \ldots, x_n) \geq (x_{k+1}, \ldots, x_n)\)

2. For any (bounded) nondecreasing function \( Y \) on \( \mathbb{R}^N \)

\[ \text{E}(Y(x_1, \ldots, x_n) | X, e[a, b], \ldots, X_n e[a, b]) \]

is weakly increasing in \((a_1, \ldots, a_n, b_1, \ldots, b_n)\)

3. \( X_1, Y_1, \ldots, Y_n \) is affiliated, where \( Y_1, \ldots, Y_{n-1} \) is the vector of order statistics for \( X_1, \ldots, X_n \).
2nd Price Auction

[Symmetric model]

- Joint Dist. of $X_1, \ldots, X_n$ symmetric

- $V_i(x) = u(x_i, x_j)$ and $u_i(x_i, \cdot)$ is symmetric in coordinates of $x_i$,
  and $u(x)$ is nondecreasing in $x$.

$V_i = V_i(x) = u(x_i, x_j)$

Define $V(x, y) = E[V_i \mid X = x, Y_i = y]$

$= E[u(x_i, y_1, \ldots, y_n) \mid X = x, Y_i = y]$ Assume $V(x, y)$ is nondecreasing in $x, y$

and strictly increasing in $x$. (Folows, for example, if $u(x_i, x_j)$ is strictly increasing in $x_i$ and $x_j$ and are affiliated.)

For 2nd price auction, symmetric joint distribution of signals and payoffs

Proposition: Let $\beta^{II}(x) = V(x, x)$; then

$(\beta^{II}, \ldots, \beta^{II})$ is a symmetric Bayes-Nash equilibrium (can show uniqueness too -- no other symmetric equilibrium -- we skip.)
Proof. We check first that \((\beta_\pi, \ldots, \beta_\pi)\) is a Bayes-Nash equilibrium. By symmetry, we can focus on buyer 1. Suppose buyer 2 through \(N\) use \(\beta_\pi\), and buyer 1 observes \(x_1 = x_1\).

What bid \(b\) is the best response for buyer 1?

Since auction is second price, buyer wins \(V_1\) if \(v(Y_1, Y_1) < b\). Suppose buyer one knew the value of \(Y_1\). He would then calculate his conditional expected payoff as

\[
E[ v(x, x_1 = x, Y_1) ] \text{ payment}
\]

\[
= \left[ v(x, Y_1) - v(Y_1, Y_1) \right] I \{ v(Y_1, Y_1) \leq b \}
\]

\[
\geq 0 \iff x \geq Y_1.
\]

So buyer 1 would be best off winning if \(x \geq Y_1\) and losing else. That is, (uniquely) accomplished by letting \(b = v(x, x) + \beta_\pi(x)\). \(\square\)
English auction - strategy

A strategy for a bidder in an English auction is an \((N-1)\)-tuple of functions

\[ \beta = (\beta_2, \ldots, \beta_n) \]  

where \(\beta_k(x, p_{k+1}, \ldots, p_N)\) is the price at which the bidder with signal \(x\) would drop out if \(k\) bidders remain and the other \(N-k\) bidders dropped out at prices \(p_{k+1} \leq \ldots \leq p_N\).

If the strategies of the other bidders are known, and if the \(\beta_k\) function of each bidder is a strictly increasing function of the bidder's value \(x_j\), then a bidder can deduce the values of those bidders that dropped out.
Assume a symmetric dependence of values on signals: \( V_i(x) = U(x_i, x_{-i}) \), where 
\( U(x_i, x_{-i}) \) is a symmetric function of \( x_{-i} \) for \( x_i \) fixed. Also, assume \( U(x_i, x_{-i}) \) is nondecreasing in \( x_i \) and strictly increasing in \( x_{-i} \).

Given \( U \), determine a strategy \( \beta \) recursively as follows:

\[
\begin{align*}
\beta^N(x) &= U(x_1, \ldots, x_N) \\
\beta^{N-1}(x, p_N) &= U(x_1, \ldots, x_{N-1}, x_N) \\
\beta^{N-1}(x, p_{N-1}, p_N) &= U(x_1, \ldots, x_{N-2}, x_{N-1}, x_N) \\
&
\vdots \\
\beta^2(x, p_3, \ldots, p_N) &= U(x_1, x_2, x_3, \ldots, x_N) \\
\beta(x, p_3, \ldots, p_N) &= U(x_1, x_2, \ldots, x_N)
\end{align*}
\]

Note: Does not involve joint pdf's.

- Choice of \( \beta \) implicitly assumes other bidders used same \( \beta^0, \ldots, \beta^N \), i.e., symmetric strategies.
Proposition. The symmetric strategy profile \((\text{determined by function } c)\) of all bidders using \(\beta\) is a Bayes-Nash equilibrium. (Even if joint distribution of the signals \(x_1, \ldots, x_n\) is not symmetric.)
Proof. Focus on bidder 1, assuming all
the other bidders follow β. Let \( y_1, \ldots, y_{n-1} \)
be a realization of the order statistics
\( Y_1, \ldots, Y_{n-1} \) based on \( X_1, \ldots, X_N \), as before.
Let \( x \) denote the value of \( X_1 \). Bidder 1
either stays in the auction till he wins, or
he drops out earlier. If he stays in his
price is \( u(y_1, y_2, y_3, \ldots, y_{n-1}) \), and his
payoff is thus \( u(x, y_1, \ldots, y_{n-1}) - u(y_1, y_2, \ldots, y_{n-1}) \).
It's positive if \( x > y_1 \) and negative if \( x < y_1 \).
Thus, if bidder 1 follows \( β \), he will win
when payoff is positive (i.e., when \( x > y_1 \)) and lose,
when payoff is negative (i.e., when \( x < y_1 \)), as
desired. \( \square \)
Q. For a given realization of \( x_1, \ldots, x_N \), does a bidder with the largest signal \( x_i \) win?

A. Yes. To see this, note that the function \( \beta^k (x_1 p_{k1}, \ldots, p_N) = u(x_1, \ldots, x_i, x_{k1}, \ldots, x_N) \) are all increasing in \( x_i \).

Q. Ultimately, the value of the object to buyer \( i \) is \( V_i (x) = u_i (x_i, x_i) \). Does a bidder with the largest value always win?

A. No, not always. For example, if

\[
V_i (x) = u(x_i, x_i) = x_i + \sum_{j \neq i} x_j = 2\left(\sum_{j=1}^{x-1} x_j \right) - x_i
\]

then the bidders with largest values have the smallest signals, and are first to drop out.
Q. Bidders drop out before they know their values. Is it possible the selling price is smaller than the value of a losing bidder?

A. Yes $N = 2$

\[ V_1(x) = x + 2x \]
\[ V_2(x) = x + 2x \]

\[ \beta(x) = U(x, x) = 3x \]

If $x_1 < x_2$, bidder 2 wins and pays $3x_1$ and has value $x_2 + 2x_1$. Bidder 1 has an even larger value, $x_1 + 2x_2$, and lost.

Q. Would a bidder ever regret not bidding higher?

A. No. In the example just given, if bidder 2 decided to win the auction, she'd have to pay $3x_1$. In general a losing buyer (say buyer 1, means $x_1 < y_1$) that deviates to win receives

\[ \text{payoff } U(x_1, y_1, \ldots, y_{n-1}) - U(y_1, y_1, \ldots, y_{n-1}) < 0 \]
Revenue Comparison (symmetric case)

\[ R^\text{II} = E[V(Y_1, Y_1) \mid Y_1 < X_1] \]

\[ R^\text{ENG} = E[U(Y_1, Y_1, Y_2, \ldots, Y_{N-1}) \mid Y_1 < X_1] \]

\[ V(y, y) = E[U(X_1, Y_1, Y_2, \ldots, Y_{N-1}) \mid X_1 = y, Y_1 = y] \]

\[ = E[U(Y_1, Y_1, \ldots, Y_{N-1}) \mid X_1 = y, Y_1 = y] \]

\[ \leq E[U(Y_1, Y_1, \ldots, Y_{N-1}) \mid X_1 = x, Y_1 = y] \text{ for any } x \geq y \]

Plugging in \( V \) in expression for \( R^\text{II} \) and comparing to \( R^\text{ENG} \) yields:

\[ R^\text{II} \leq R^\text{ENG} \]

Note: There are all equalities except at (*)

Increasing the conditioning value for \( X_1 \) causes (likelihood ratio ordering) increase in \( (Y_2, \ldots, Y_{N-1}) \). These stochastically larger values are observed/plugged in, in case of English auction essentially letting seller capture the gain.