Extensive Form Games

Nim

Start with three piles, sizes 5, 4, 3
Play alternates between players
during turn, player removes one or more sticks from one pile
player to remove last stick loses

demo a few times

How to model?

A state \( r = (r_1, r_2, r_3) \) \( r_i \) # sticks remaining in pile \( i \)

Label non-terminal nodes by player, inf-state
terminal nodes by payoff vectors
edges (branches) by actions
Nim has perfect information for each player. Same as chess or checkers. In principle, can solve such games by backwards induction (or permutations of states shown)

<table>
<thead>
<tr>
<th>Winning states</th>
<th>Losing states</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2+)00, (1+)10</td>
<td>100</td>
</tr>
<tr>
<td>(2+)11, (3+)20</td>
<td>111, 220</td>
</tr>
<tr>
<td>32(+)</td>
<td>321, 330</td>
</tr>
<tr>
<td>(3+)31, (4+)21 (4+)30, 33(+)</td>
<td>440</td>
</tr>
<tr>
<td>540, 44(+)</td>
<td>541</td>
</tr>
</tbody>
</table>

Player 1 can always win the 543 game.

Theory of nim sums gives general solution. Decompose each pile uniquely into subpiles with powers of two sticks, and cancel subpiles from different piles.
Call My Bluff Card game
(Myerson, Chap. 2)
A chance mechanism selects red or black card
(equally likely) - player 1 observes but not
player 2. Red card is good for player 1.

One strategy pair is for
player one to raise on red and
fold on black "Rf"
Then optimal response for player 2
is always pass. Mean value is 0,0
Can either player do better?
The normal representation of an extensive form game is the strategic form game obtained by having each player select a table of contingencies -- one for each of the information states at nodes under control of the agent.

For the call my bluff card game, this means player one decides which action to take for a red card (i.e. Raise or Fold) and for a black card (i.e. Raise or Fold). Player 2 Meets or Passes. The payoffs are given by averaging over the leaf nodes:

\[ U_i(c) = \sum_{x \in \Omega^*} P(x|c) w_i(x) \]

player vector of actions for all information states
The normal representation of the call my bluff card game:

<table>
<thead>
<tr>
<th></th>
<th>M</th>
<th>P</th>
</tr>
</thead>
<tbody>
<tr>
<td>Rr</td>
<td>0,0</td>
<td>1,1</td>
</tr>
<tr>
<td>RF</td>
<td>5,5</td>
<td>0,0</td>
</tr>
<tr>
<td>Ff</td>
<td>0,0</td>
<td>0,0</td>
</tr>
<tr>
<td>Fr</td>
<td>-5,-5</td>
<td>1,1</td>
</tr>
</tbody>
</table>

\[ A_1(RF, M) = \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot (-1) = 0.5 \]
\[ A_1(RF, P) = \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1) = 0 \]

Let's analyze this static game.

Any strongly dominated strategies? No.
weakly  "  yes Fr, Ff are weakly dominated by Rr

pure strategy NE? no.

Suggests \((0, 1-a, 0, 0) \geq (6, 1-b)\)
\[ 0 < a \leq 1 \]
\[ 0 < b < 1 \]
\[ (0.5)(1-a) = a \]
\[ b = \frac{2}{3} \]

NE (saddlepoint): \((\frac{1}{3}, \frac{2}{3}, 0, 0), (\frac{2}{3}, \frac{2}{3})\)

- Value (player 1) = \(0.4 \cdot 0 + \frac{1}{3} \cdot 1 + \frac{4}{9} \cdot 5 + \frac{2}{9} \cdot 0 = \frac{1}{3} \)

Player 1 always raises on red, raises on black w/p \(\frac{1}{3}\).
Player 2 meets with prob. \(\frac{2}{3}\).
A game in strategic form can be represented as a game in extensive form:

Example: matching pennies

![Game Tree](attachment:image.png)

Information state A (⇒) player 1 knows game is at the beginning.

Information state B (⇒) player 1 has selected move but value is unknown.

Dashed circle indicates grouped nodes have same information state.
If we change the information structure of the previous game:

Then the normal form of the game becomes:

<table>
<thead>
<tr>
<th></th>
<th>H</th>
<th>T</th>
</tr>
</thead>
<tbody>
<tr>
<td>H</td>
<td>1,1</td>
<td>1,1</td>
</tr>
<tr>
<td>T</td>
<td>1,1</td>
<td>1,1</td>
</tr>
</tbody>
</table>

Player 2:

- Weakly dominant for player 2 - always wins.
Extensive form vs: normal representation 

(normal representation *) (Myerson Fig. 2.6)

Normal representation (a strategic form game)

\[
\begin{pmatrix}
\text{Player 1} & y_2 & 2_2 \\
\text{a}_1, w_1 & 5.0 & 3.0 \\
\text{b}_1, w_1 & 3.3 & 3.0 \\
\text{b}_1, x_1 & 7.3 & 4.0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\text{Player 2} & y_2 & 2_2 \\
\text{w}_1 & 5.0 & 3.0 \\
\text{x}_1 & 4.0 & 4.0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\text{Agent 1.1} & 6_1 \\
\text{w}_1 & 5.5 & 5.0 \\
\text{x}_1 & 4.3 & 7.0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\text{Agent 1.2} & 6_2 \\
\text{w}_1 & 5.5 & 5.0 \\
\text{x}_1 & 4.3 & 7.0 \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
\text{Agent 1.1} & 6_1 \\
\text{w}_1 & 5.5 & 5.0 \\
\text{x}_1 & 4.3 & 7.0 \\
\end{pmatrix}
\]

Replace player 1 by agents 1.1 and 1.2, each getting same payoff as player 1.

\[(6_1, y_2, w_1) \text{ is a NE}\]
Definition A behavioral strategy profile is any randomized-strategy profile for the multiaagent representation of $\Pi^c$.

Mixed strategy profiles

$\times \Delta(C_i)_{i \in \text{players}}$

Behavioral-strategy profiles

$\times \times \Delta(D_s)_{i \in \text{players}, \text{sets}}$

information states for player $i$

decision set for information states
Example (Myerson Fig. 4.1, p. 157)

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Normal representation

Player 2

<table>
<thead>
<tr>
<th></th>
<th>( w_1 ), ( y_1 )</th>
<th>( w_2 ), ( y_2 )</th>
<th>( x_1 ), ( y_2 )</th>
<th>( x_2 ), ( y_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( w_1 ), ( y_1 )</td>
<td>3, 1</td>
<td>2, 2</td>
<td>3, 1</td>
<td>1, 3</td>
</tr>
<tr>
<td>( w_2 ), ( z_1 )</td>
<td>2, 2</td>
<td>3, 1</td>
<td>1, 3</td>
<td>2, 2</td>
</tr>
<tr>
<td>( x_1 ), ( y_1 )</td>
<td>3, 2</td>
<td>1, 3</td>
<td>3, 1</td>
<td>2, 2</td>
</tr>
<tr>
<td>( x_2 ), ( z_1 )</td>
<td>1, 3</td>
<td>2, 2</td>
<td>2, 2</td>
<td>3, 1</td>
</tr>
</tbody>
</table>

Game is equivalent to first selecting one of the two games using a coin flip:

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```
```

Both are variations of matching penny's game.

Both are variations of matching penny's game.
The equilibrium behavioural-strategy profile is:

\[
\left( 0.5[w_1] + 0.5[x_1], 0.5[y_1] + 0.5[z_1], 0.5[w_2] + 0.5[x_2], 0.5[y_2] + 0.5[z_2] \right)
\]

(agent for player 1)

(agent for player 2)

(Much simpler)

The equilibria in mixed strategy profile form are, for \( 0 < \alpha < 1/2 \) and \( 0 < \beta < 1/2 \)

\[
\left( \alpha [w_1, y_1] + \alpha [x_1, z_1] + (1-\alpha) [w_2, z_1] + (1-\alpha) [x_1, y_1], \\
\beta [w_2, y_2] + \beta [x_2, z_2] + (1-\beta) [w_2, z_2] + (1-\beta) [x_2, y_2] \right)
\]

\( \alpha \) gives correlation between how player 1 would play in two subgames.

... largely irrelevant, eg. unobservable from multiple plays

\( \beta \) similarly
Example of computing a behavioral representation

(Myerson Fig 4.2 p. 158)

\( Z_1 = (0.5 q_1 y_1 + 0.5 b_1 z_1) \) for player 1

What behavioral representation of \( Z_1 \)?

\[ \psi_1 = (\psi_{1,1}, \psi_{1,2}) = (0.5, 0.5), (1, 0) \]

\[ = (0.5 q_1 + 0.5 b_1, [y_1]) \]

Do we simply want equilibrium for multi-agent representation?

How about \( ([b_1], [w_2], [2, 3]) \) - payoff \( (2, 2) \).

Player 1 would like \([q_1], [y_1]\) better.
Normal form of this game:

$$\begin{array}{c|cc}
 & x_1 & x_2 \\
\hline
a_1 & 3 & 2 \\
a_2 & 0 & 5 \\
b_1 & 3 & 3 \\
b_2 & 3 & 3 \\
\end{array}$$

If player 1 restricts to \(a_1, y_1, b_1, y_1\), looks like matching pennies.

\((0.5[a_1] + 0.5[y_1], 0.5[w_2] + 0.5[x_2])\)
\((0, 5, 0, 5)\) is a NE for the normal form.

\[
\begin{bmatrix}
(0.5[a_1] + 0.5[y_1], 0.5[w_2] + 0.5[x_2], [y_1])
\end{bmatrix}
\]

is the behavioural representation of the mixed NE.

(Myerson p. 161) FrT is similar but not explicit.

Definition: A Nash equilibrium of an extensive form game (with perfect recall) (aka equilibrium in behavioral strategies) is a behavioral strategy profile \(\sigma\) such that (1) \(\sigma\) is a NE at the multiagent representation of the game and (2) the mixed representation of \(\sigma\) is a NE of the normal representation.
Perfect recall condition

An extensive form game satisfies the perfect recall condition if whenever a player moves, he remembers all information he knew at earlier positions.

More formally, perfect recall condition is that if nodes x, y, z exist as in this picture, a node w must also exist as in picture, with outgoing branch of same label as shown. (Fig 9.2, p. 160 Myerson)

Example violating perfect recall: (violates Kuhn's theorem too)
Behavioral representation

Consider an extensive form game

and a player $i$. A mixed strategy for player $i$ at the normal form game given by $\tau_i \in \Delta(C_i)$. Such a $\tau_i$ has a behavioral representation $\tau_i \in \mathcal{X} \times \Delta(D_s)$, required to satisfy:

$$\mathcal{X}_{i,s}(D_s) \left( \sum_{e_i \in C_i^*(s)} \tau_i(e_i) \right) = \sum_{c_i \in C_i^{**}(d_s)} \tau_i(C_i)$$

really only makes sense if the perfect recall condition is satisfied.

- Surprising: this is enough: behavioural representation is defined not depending on play of other players.
Kuhn's Theorem (Kuhn 1953) If \( T^c \) is an extensive form game with perfect recall, for any player \( i \), if \( \sigma_i \) is a behavioral representation of a mixed strategy \( \pi_i \), for any profile of strategies for the other players \( \pi_{-i} \), all players have the same (expected) payoffs under \((\sigma_i, \pi_{-i})\) and \((\pi_i, \pi_{-i})\).

Corollary. If \( T^c \) is an extensive form game with perfect recall, if \( \sigma \) is a behavioral strategy profile and \( \pi \) a mixed strategy profile of the normal representation of \( T^c \), and \( \sigma_i \) is a behavioral representation for \( \pi_i \) for all \( i \), then \( \sigma \) is an NE of \( T^c \) iff \( \pi \) is an NE of the normal representation of \( T^c \).

Corollary. NE exists for extensive game with perfect recall.
Entry Deterrence Game
Player 1 = entrant, Player 2 = incumbent

```
1.a  in
\_\_\_
  out (2,1)

out (0,0)
```

Normal representation

```
\begin{array}{c|cc}
       & \text{fight} & \text{accom} \\
\hline
\text{in}  & 2,1          & 0,0   \\
\text{out} & 1,2          & 1,2   \\
\end{array}
```

Of the two NE, only (2,1) is subgame perfect.

A (proper) subgame is defined by an information state \( s \) consisting of a single node \( x \), so that for any node \( y \) following \( x \), all nodes with the same information state as \( y \) also follow \( x \). A strategy profile \( \sigma \) is subgame perfect if \( \sigma_i \) is a best response to \( \sigma_{-i} \) for any (proper) subgame.
If we slightly modify entry deterrence game,

we can get:

\[
\begin{array}{ccc}
 & A & 2,1 \\
L & F & 0,0 \\
\end{array}
\]

\[\begin{array}{ccc}
in_1 & A & 2,1 \\
in_2 & F & 0,0 \\
\end{array}\]

A = accommodate
F = fight

Note that actions in \(i\) and \(i_{n_2}\) are equivalent. But now there is no subgame for player 2 beginning at a node. So \((\text{out}, F)\) is subgame perfect.

To rule it out, Kreps and Wilson (\'82) defined sequential equilibrium in which games starting at information states with multiple nodes are considered, (as well as subgames from single node information states).
Sequential equilibria (strengthening of subgame perfect equilibria)

Recall that in an extensive form game, an information state $s$ is a set of nodes grouped by dashed lines. It is a maximal set of nodes with the same information state.

All the nodes in an information state are controlled by a single player. A behavioral strategy profile $\sigma = (\sigma_i)$ consists of a probability distribution $(\sigma_i(s) : s \in D_s)$ over the decision set $D_s$ for every player $i$ and every information state $s$ for $i$.

A belief vector $\mu$ consists of a probability distribution $(\mu(x|s) : x \in s)$ of $s$ for every information state $s$.

An pair $(\sigma, \mu)$ consisting of a behavioral strategy profile $\sigma$ and a belief vector $\mu$ is called an assessment.
For an extensive form game with imperfect information, payoffs are determined by the terminal node that is reached.

It is therefore easy to define:

\[ u_i(\sigma | x) \] (payoff for player \( i \) given node \( x \) is reached.)

Such payoffs only depend on decisions made at \( x \) and later in the game. For an information state \( s \) and belief vector \( \mu(s) \), we define

\[ u_i(\Delta | s, \mu(s)) = \sum_{x \in S} u_i(\sigma | x, \mu(x | s)) \]
Definitions

(s) An assessment \((\sigma, \mu)\) is sequentially rational if, for any player \(i\), any information state \(s\) of player \(i\), and any alternative behavioural strategy \(\bar{\sigma}_i\) for player \(i\),
\[ u_i(\sigma | s, \mu(s)) \geq u_i((\sigma'_i, \bar{\sigma}_i) | s, \mu(s)) \]

Let \(\Sigma^0_i\) be the set of fully mixed behavioral strategies, and let \(\Psi^0 = \{ (\sigma, \mu) : \sigma \in \Sigma^0_i, \mu_{s,s}(x) = \frac{p(x | s)}{\sum_{x'} p(x' | s)} \}\)

(c) An assessment \((\sigma, \mu)\) is consistent if there exists \((\sigma^{(n)}, \mu^{(n)})\) \(\in \Psi^0\) with
\[ (\sigma, \mu) = \lim_{n \to \infty} (\sigma^{(n)}, \mu^{(n)}) \]

Def: A sequential equilibrium is an assessment \((\sigma, \mu)\) satisfying \(s\) and \(c\).
Def: \(\sigma\) is a sequential-equilibrium scenario if \((\sigma, \mu)\) is a sequential equilibrium for some \(\mu\).
(Setting: Extensive form game with perfect recall.)

**Proposition** If \( \sigma \) is a (trembling hand) perfect equilibrium of the multi-agent representation of \( \Gamma^e \), then \( \sigma \) is a sequential-equilibrium scenario.

(Myerson p. 217)

**Proof** By definition, there is a sequence \( (\sigma^{(n)}) \) of fully mixed profiles with \( \sigma^{(n)} \rightarrow \sigma \) and \( \sigma_i \in B(\sigma^{(n)}_i) \) for all \( n \). Let

\[
\mu_i^{(n)}(x) = \frac{\rho(x\mid \sigma^{(n)})}{\sum_{x' \in \Omega} \rho(x'\mid \sigma^{(n)})}
\]

(i.e. \( \mu^{(n)} \) is the unique belief vector determined by \( \sigma^{(n)} \) and Bayes’ rule for each information state \( s \).)

By compactness, going to a subsequence if necessary, we can assume \( \lim_{n \to \infty} \mu^{(n)} \) exists. Let limit be \( \mu \).

Then \( (\sigma, \mu) \) is a sequential equilibrium (15) following using continuity of \( u_i \) and

letting \( n \to \infty \) in

\[
u_i((\sigma^e, \sigma^{(n)}_i) 1_s, \mu^{(n)}(s)) = \nu_i((\sigma^e, \sigma^{(n)}_i) 1_s, \mu^{(n)}(s))
\]

for any behavioral profile \( \sigma^e \).
Example illustrates condition in previous proposition.

Extensive form

\[
\begin{array}{c|cc}
 & 1, a & 2, b \\
\hline
U & 4, 4 & 4, 4 \\
D & 1, 1 & 2, 2 \\
\end{array}
\]

\[
\begin{array}{c|cc}
 & 1, a & 2, b \\
\hline
W & 4, 4 & 4, 4 \\
Z & 1, 1 & 2, 2 \\
\end{array}
\]

(Uu, Z) is a trembling hand perfect equilibrium of the normal representation of the game. While \( \sigma \) defined by \( \sigma = ([U], [Z], [U]) \) is a behavioral representation of \( \Gamma \), it is only an NE, not a THP equilibrium of the multiagent version of the game. Problem is, under Uu, player 1 doesn't recognize possibility he trembled at state 1. a when considering 1. c.

Note: ([U], [Z], [d]) is THP for multiagent game.
Note: If an extensive form game has the property that no player can play twice (or more) in any instance of the game, then if $T$ and $T'$ are mixed strategies for the normal form game both equivalent to the same behavioural strategy $T$, then there is no way to distinguish among $T$, $T'$, or $T$ when watching the game. Moreover, $T$ is THP equilibrium for the extensive form representation of the game if and only if $T$ is a THP equilibrium for the multiagent representation. If $T$ is THP, $T$ is a sequential-equilibrium scenario.
Proposition (Selton '75) A finite strategic form game \((I, (A_i; i \in I), (u_i; i \in I))\) has at least one trembling-hand perfect equilibrium.

Corollary Every finite extensive form game has at least one sequential-equilibrium-scenario \(\Sigma\).

Notation For a finite set \(F\) and \(\varepsilon > 0\) with \(|F| \leq 1\), let \(\Delta_\varepsilon(F)\) denote the set of probability vectors \(p\) over \(F\) with \(p(a) \geq \varepsilon\) for all \(a \in F\).

Proof of proposition For \(\varepsilon > 0\) small enough, consider the \(\varepsilon\)-perturbed game with payoff functions 

\[ u_i(t_i, t_{-i}) \text{ for } t_i \in \Delta_\varepsilon(A_i). \]

Let \(\varepsilon_n \to 0\). For each \(n\) sufficiently large there is a NE profile 

\(\Sigma^{(n)}\) for the \(\varepsilon_n\)-perturbed game (by fixed point theorem as used to prove Nash's theorem).
By compactness, going to a subsequence of \( n \) if necessary, it can be assumed \( \lim_{n \to \infty} t_i^{(n)} \) exist. Let \( t_i^{(n)} = \lim_{n \to \infty} t_i^{(n)} \). Note that \( t_i = (t_i)_{i \in I} \) is a mixed strategy profile. Let \( A_i^* = \{ a_i \in A_i : t_i(a_i) > 0 \} \), i.e., \( A_i^* \) is the support set for \( t_i \). By going to a subsubsequence if necessary, it can be assumed \( t_i^{(n)}(a_i) > \epsilon^{(n)} \) for all \( i \in I \), all \( a_i \in A_i^* \), and all \( n \).

Thus, \( A_i^* \subseteq B_i(t_i^{(n)}) \) for all \( i \in I \) and all \( n \). Therefore, \( t_i \) is a best response to \( t_i^{(n)} \) for all \( i, n \), and \( t^{(n)} \) is a profile of fully mixed strategies with \( t^{(n)} \rightarrow t \). So \( t \) is trembling hand perfect.

\[ \square \]

Problem: Find a polynomial time algorithm to compute a trembling hand perfect equilibrium for a two person zero-sum game.
Proposition. In a two player matrix game, a strategy profile $T = (T_1, T_2)$ in mixed strategies is THP if and only if neither $T_1$ nor $T_2$ is weakly dominated (here $p$ weakly dominates $p'$ for a player $i$ if $U_i(p, j) \geq U_i(p', j)$ with strict inequality for at least one value of $j$.) (See, e.g. Osborne & Rubinstein)