University of Illinois **Spring 2013**

ECE 586BH: Problem Set 6

Revenue optimal selling mechanisms, auctions with interdependent values

Due: Thursday, April 25 at beginning of class
Reading: V. Krishna, *Auction Theory,* Chapter 5 (based largely on Myerson 1982 paper) and Chapter 6 (based on Milgrom and Weber 1983 paper)

1. **[VCG example - sale of two items]**
Suppose a VCG mechanism is applied to sell the objects in $\mathcal{O} = \{a, b\}$ to three buyers. A buyer can buy none, one, or both of the objects. For simplicity, assume the valuation function of each buyer depends only on the set of objects assigned to that buyer. The values are:

$$u_1(\emptyset) = 0,\ u_1(\{a\}) = 10,\ u_1(\{b\}) = 3,\ u_1(\{a,b\}) = 13$$
$$u_2(\emptyset) = 0,\ u_2(\{a\}) = 2,\ u_2(\{b\}) = 8,\ u_2(\{a,b\}) = 10$$
$$u_3(\emptyset) = 0,\ u_3(\{a\}) = 3,\ u_3(\{b\}) = 2,\ u_3(\{a,b\}) = 14$$

To be definite, suppose the standard version of the payment rule is used whereby for each buyer i, $m_i(\widehat{u})$ is the maximum welfare of the other buyers minus the realized welfare of the other buyers, both computed using the reported valuation functions.

   (a) Determine the assignment of objects to buyers and the payments of the buyers, under truthful bidding.

   (b) Discuss why buyer 3 might have an objection to the outcome.

2. **[About the virtual valuation functions in revenue optimal seller mechanisms]**
Given a pdf f_i with support equal to the interval $[0, \omega_i]$, the virtual valuation function is defined on the same interval and is given by $\psi_i(x_i) = x_i - \frac{1-F_i(x_i)}{f_i(x_i)}$.

   (a) Show that $E[\psi_i(X_i)] = 0$.

   (b) Find ψ_i in case X_i is exponentially distributed with parameter $\lambda > 0$.

   (c) Suppose ψ_i is an increasing function on $[0, \omega_i]$. Work backwards to express the CDF F_i in terms of ψ_i, being as explicit as possible. To be definite, suppose $F_i(0) = 0$. What additional assumptions are needed on ψ so that F_i is nondecreasing with $F(\omega_i) = 1$?

3. **[A mechanism to maximize a mixed objective function]**
Suppose N buyers have known independent valuations with the valuation of buyer i having strictly positive pdf f_i over the interval $[0, \omega_i]$. As discussed in class, the revelation principle and revenue equivalence principle can be used to help identify a simple structure for the (revenue) optimal auction. The revelation principle can also be used to help derive a maximum welfare auction; the auction turns out to be the second price auction. For this problem, suppose α is fixed with $0 \leq \alpha \leq 1$. Find a direct mechanism (Q, M) with truthful equilibrium that is individually rational, which, at the equilibrium point, maximizes $\alpha\times$(revenue to seller) + $(1 - \alpha) \times$ (social welfare). By the revelation principle, there is no loss of optimality in restricting attention to such mechanisms. (Hint: Follow the derivation of the revenue optimal auction.)

(a) Does the revenue equivalence principle apply? If so, what can be deduced from it?

(b) Propose a selection rule for the auction.

(c) State a regularity assumption similar to the one made about the functions ψ_i for the revenue optimal auction, which insures the selection rule proposed in part (b) is optimal for the mixed objective function of this problem. Finally, identify the corresponding payment rule, that reduces to the second price payment rule in the special case $\alpha = 0$.

4. **[On some properties of (revenue) optimal auctions]**
Consider the basic revenue optimal direct mechanism with truthful equilibrium, assuming N buyers with independent valuations, with the valuation of buyer i known to have pdf f_i with support equal to the interval $[0, \omega_i]$, for each i. Assume regularity, i.e. the virtual payoff functions ψ are strictly increasing.

   (a) The function $m_i(x_i)$ determines only the expected payment of buyer i, given buyer i reports x_i. A particular form of the actual payment is $M_i(x_i) = y_i(x_{-i})I_{\{i \text{ wins}\}}$, where

   $$y_i(x_{-i}) = \min\{z_i : \psi_i(z_i) \geq 0 \text{ and } \psi_i(z_i) \geq \psi_j(x_j), j \neq i\},$$

   Truthful reporting is a Bayes-Nash equilibrium. Is truthful reporting for a given buyer i also a weakly dominant strategy? Explain your answer.

   (b) For the payment rule in part (a), is it possible that for buyer i and some realization of the vector X, buyer i bids truthfully and is a winner, but his payment is strictly larger than X_i?

   (c) Another form for the payment rule is $M_i(x) = P_i(x_i)I_{\{i \text{ wins}\}}$, where the function P_i is selected so that $E[M_i(X)|X_i = x_i] = m_i(x_i)$, where m_i is the same function as before, determined by incentive compatibility. This rule has the advantage for player i that if he wins, he will know what his payment will be, no matter what the other buyers bid. Express the function P_i in terms of the function q_i. (Truthful reporting should be a Bayes-Nash equilibrium.) Is truthful reporting a weakly dominate strategy for each buyer?

5. **[Illustration of need for affiliation assumption]**
A random vector $X = (X_1, X_2, X_3, X_4)$ is generated as follows. A fair coin is flipped, and if heads shows, $X = \Pi(1, 0.9, 0.9, 0.9)$, and if tails shows, $X = \Pi(1, 1.3, 0, 0)$, where Π is a random permutation acting on 4-tuples, with all 4! possibilities having equal probability. Let $V = X_1 + X_2 + X_3 + X_4$. Note that the distribution of X is symmetric and V is a symmetric increasing function of X. Find and compare $E[V|X_1 = 1]$ and $E[V|X_1 = 1, Y_1 < 1]$, where $Y_1 = \max\{X_2, X_3, X_4\}$. (This shows that a statement at the top of p. 85 of Vijay Krishna's book needs an assumption such as affiliation of the X_i's.)

6. **[An example with interdependent values]**
Let the signals $X_1, ..., X_N$ be independent, uniformly distributed on the interval $[0, 1]$, and let the values be given by $V_i = \alpha X_i + \sum_{j=1}^{N} X_j$, where α is a known constant in $(-1, \infty)$. (In the following you may use the fact that N independent random variables that are uniformly distributed on the interval $[0, 1]$ divide the interval into $N+1$ random subintervals, the lengths of which are identically distributed. Consequently, for example, $E[Y_1|Y_1 < X_1] = \frac{N-1}{N+1}$, because given $Y_1 < X_1$, Y_1 is the second largest of n independent uniformly distributed random variables.)

(a) Assuming a second price auction, identify the symmetric Bayes-Nash equilibrium strategy and the resulting average revenue to the seller.

(b) Assuming an English auction, calculate the average revenue to the seller, and compare to the revenue for the second price auction.