University of Illinois **Spring 2013**

ECE 368BH: Problem Set 3

Analysis of static games (continued), and dynamics involving static games

Due: Thursday, March 7 at beginning of class

Reading: Menache and Ozdaglar, Part I (pp. 63-67 introduces potential games).
Material on players based on minimizing regret, and the Blackwell approachability theorem, can be found in Cesa-Bianchi and Lugosi, *Prediction, Learning, and Games*, Chapters 2,4, and 7.
Students at UIUC can access eBook–see link on course webpage

1. **[Two player Cournot game]**
Consider the two player Cournot game with actions $s_i \in [0, \infty)$ and payoff function $\pi_i(s) = (a-(s_1+s_2)-c)s_i$. Here s_i is the amount produced by player i, the cost per unit of production is c, and the price for total quantity $s_1 + s_2$ is $a - s_1 - s_2$. Assume $0 < c < a$. Show that the game is solvable by iterated elimination of strongly dominated strategies. (Infinitely many iterations are required. To get some intuition, sketch $\pi_1(s_1, s_2)$ as a function of s_1, for various values of s_2.)

2. **[Two-player, two-action, static potential games]**
Consider the two-player, two-action, static potential game shown.

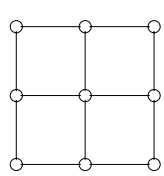
	1	2
1	a,b	c,d
2	e,f	g,h

Under what condition on the constants a through h is there an exact potential function $(\Phi(i,j) : i,j \in \{1,2\})$ for this game? Give a potential function in case the condition holds.

3. **[A simple graphical congestion game.]**
Let G be an undirected graph, $G = (V, E)$, where V is the set of vertices and E is the set of edges. Suppose the graph represents a campground, with V denoting the players, who are campers, and an edge $e = [i, j]$ connecting two players if their campsites are neighboring. Each player $i \in V$ has a decision variable $s_i \in S_i = \{0, 1\}$, where $s_i = 1$ means the player will play music and s_i means the player will not play music. The value for playing music to player i is v_i, but if a player plays music she also suffers a congestion cost equal to the number of neighboring campers (not including i) who play music. Thus, the payoff for player i is given by $\pi_i(s) = s_i \left(v_i - \sum_{j \in \mathcal{N}(i)} s_j\right)$, where $\mathcal{N}(i)$ is the set of neighbors of i, not including i.

   (a) Show that the congestion game $(V, (S_i, i \in V), (\pi_i : i \in V))$ has an exact potential function Φ, and give a simple expression for it.

   (b) Illustrate the convergence of best response dynamics for the graph shown, assuming $v_i = 1.5$ for all i.



Begin with the all zero strategy vector, and cycle through the players one at a time, first visiting players in the top row left to right, then the second row, and so on.

4. **[Exponential regret strategy vs. an oblivious player for rock, paper, scissors game]**
Consider two players playing the zero sum rock (R) , paper (P), scissors (S) game, for $n = 1000$ rounds. Focus on the cumulative loss of player 1, in the case player 2 plays R for the first 500 rounds and then P for the remaining 500 rounds. (Player 2's strategy is far from being a stationary one.)

   (a) Describe how you think you would do against player 2, if all you knew to begin with is that you were to play 1000 times, and you saw the outcome after each play. (This is highly subjective–there is no single correct answer.)

   (b) Identify the cumulative loss functions $L_{R,t}, L_{P,t}, L_{S,t}$ for constant play R, P, or S by player one. (Note: For example, $L_{P,t} = -t$ for $0 \le t \le n/2$ because playing P wins each of the first 500 games, or equivalently, results in loss -1 in each of those games.)

   (c) Suppose player one plays the exponential weighted strategy: $\widehat{p}_{t+1}(i) = \frac{e^{-\eta L_{i,t}}}{D_t}$ for $i \in \{R, P, S\}$, where $\eta = 0.1$ (which is near $\sqrt{8(\ln 3)/1000}$), and D_t is the normalizing constant making $\widehat{p}_{t+1}$ a probability distribution. Show that the expected number of times player one wins is less than or equal to 501.

5. **[Predictor based on experts derived using a quadratic potential function]**
Suppose the loss function $l(p, y)$ is a convex function of p with values in $[0, 1]$. A predictor $(\widehat{p}_t : 1 \le t \le n)$ is to be used such that $\widehat{p}_t$ is a weighted average of the predictions $(f_{i,t} : 1 \le i \le M)$ produced by M experts. The cumulative loss functions for the forecaster and experts are given by $\widehat{L}_t = \sum_{1 \le s \le t}^{t} l(\widehat{p}_s, y_s)$ and $L_{i,t} = \sum_{1 \le s \le t} l(f_{i,s}, y_s)$, respectively. The (cumulative) regret vector at time t is defined by $R_t = (\widehat{L}_t - L_{i,t} : 1 \le i \le M)$. Use the potential function $\Phi(x) = \sum_{i=1}^{N} ((x_i)_+)^2$.

   (a) Derive the expression for $\widehat{p}_t$ as a function of $(y_s, \widehat{p}_s, (f_{i,s} : 1 \le i \le M) : 1 \le s \le t-1)$ using the weight vector $w_{t-1} = \nabla\Phi(R_{t-1})$. (If all the weights are zero, let $\widehat{p}_t$ be an arbitrary element of the prediction space.)

   (b) Show that $\Phi(R_t) \le \Phi(R_{t-1}) + M$ for all $t \ge 0$. (Hint: Treat the case $\Phi(R_{t-1}) = 0$ separately. You may also use the fact that if g is a continuously differentiable function on the line with a piecewise continuous second derivative, then
   $g(b) \le g(a) + g'(a)(b-a) + \sup_{\eta \in [a,b]} \frac{g''(\eta)(a-b)^2}{2}$.)

   (c) Show that $\max_i R_{i,n} \le \sqrt{nM}$ for $n \ge 1$.

6. **[Simulation of guessing within one]**
For this problem you are to write another computer simulation for the game considered in problem 5 of problem set 2. This time you are to simulate the case that both players use the exponentially weighted forecaster with time-varying parameter $\eta_t = \sqrt{8(\ln(6)/t}$ for generation of plays at time $t+1$ for all $t \ge 1$. This value is suggested by Corollary 4.3 in Cesa-Bianchi and Lugosi. (Note that if instead we used $\eta_t = \frac{1}{\gamma t}$ we would be using exactly the soft max rule for fictitious play.) To be definite, suppose both players would like to maximize their

payoffs, where strategy sets are $\{1, 2, 3, 4, 5, 6\}$ and the payoff matrices are

$$A_1 = \begin{pmatrix} 110000 \\ 111000 \\ 011100 \\ 001110 \\ 000111 \\ 000011 \end{pmatrix}$$

and $A_2 = ones(6, 6) - A_1$. In this context, the instantaneous regret of a player for strategy i at time t is the payoff for strategy i against the play of the other player at time t, minus the payoff of the player for time t. As we know from problem set 2, the maxmin probability of winning for either player for this game is 0.5. Thus, if player one uses a mixed strategy p, the gap from optimality of p is $gap_1(p) = 0.5 - \min\{pA_1\}$. Similarly, the gap from optimality of a strategy q for player two is $gap_2(q) = 0.5 - \min\{qA_2\}$. At each time t you will need to first compute the mixed strategies for the two players, and then generate the actually (pseudo) random plays. The following matlab function could be useful.

```
function x=distRand(p)
% Generates a random variable in {1, ... , length(p)} using probability vector p
k=length(p);
p=reshape(p,k,1);
x=sum(repmat(rand(1,1),k,1) > repmat(cumsum(p)/sum(p),1,1),1)+1;
end
```

Here is what you are to turn in: (a) a copy of your computer code, (b) for $n = 100$, give

- the 6×6 matrix with i, j^{th} entry equal to the number of times player one played i and player 2 played j over times $1 \leq t \leq n$,
- the empirical distribution of plays for each player and the gap from optimality for those distributions, at time n.

and (c) same as (b) for $n = 1000$.

7. **[Approachability for a simple additive game]**
Consider the following two player game with strategy spaces $S_1 = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$ and $S_2 = \{y \in \mathbb{R}^2 : ||y|| \leq r\}$, where r is a positive constant and $||\ ||$ is the usual Euclidean norm. The payoff for player one is the vector $x + y$. Thus, the payoff is a vector, rather than a single number, as considered by Blackwell. You may assume Blackwell's approachability theorem applies to this problem. (A simple modification of the proof for the case of finite games can be used to prove it.)

   (a) An arbitrary half space H in the plane can be expressed as $H = \{x : x \cdot u \leq c\}$, where u is a unit length vector in $\mathbb{R}^2$ and $c \in \mathbb{R}$. Under what condition on u and c is this half space approachable?

   (b) Assuming that $r \leq 1$, find a simple necessary and sufficient condition for a closed set $A \subset \mathbb{R}^2$ to be approachable (for player one). Your answer should be simpler than just that all half spaces containing A should be approachable. (Hint: For what p is the singleton set, $\{p\}$, approachable. Also, think about what is approachable for player 2.)

   (c) Assuming that $r > 1$, find a simple necessary and sufficient condition for a set $A \subset \mathbb{R}^2$ to be approachable (for player one). Your answer should be simpler than just that all half spaces containing A should be approachable.