University of Illinois **Spring 2013**

ECE 368BH: Problem Set 1
Analysis of static games

Due: Thursday, January 31 at beginning of class
Reading: Menache and Ozdaglar, Part I

1. **[Guessing 2/3 of the average]**
Consider the following game for n players. Each of the players selects a number from the set $\{1,\dots,100\}$, and a cash prize is split evenly among the players who's numbers are closest to two-thirds the average of the n numbers chosen.

   (a) Show that the problem is solvable by iterated elimination of *weakly* dominated strategies, meaning the method can be used to eliminate all but one strategy for each player, which necessarily gives a Nash equilibrium. (A strategy μ_i of a player i is called weakly dominated if there is another strategy μ_i' that always does at least as well as μ_i, and is strictly better than μ_i for some vector of strategies of the other players.)

   (b) Give an example of a two player game, with two possible actions for each player, such that iterated elimination of weakly dominated strategies can eliminate a Nash equilibrium.

   (c) Show that the Nash equilibrium found in part (a) is the unique mixed strategy Nash equilibrium (as usual we consider pure strategies to be special cases of mixed strategies). (Hint: Let k^* be the largest integer such that there exists at least one player choosing k^* with strictly positive probability. Show that $k^*=1$.)

2. **[A game for allocation proportional to bid]**
Suppose an amount C of a divisible resource such as communication bandwidth is to be allocated to n buyers. Each buyer i submits a positive bid, b_i, and the vector of all bids is denoted by $\mathbf{b}=(b_1,\dots,b_n)$. In return, the buyer pays the amount b_i and receives an amount $x_i=\frac{Cb_i}{B}$ of the resource, where $B=b_1+\cdots+b_n$. The payoff for buyer i is $\pi_i(\mathbf{b})=U_i(x_i)-b_i$, where U_i is a concave, continuously differentiable function on $(0,\infty)$ with $\lim_{x\to 0}U_i'(x)=+\infty$ and $U_i'(x)>0$ for $0<x_i\le C$.

   (a) Characterize the value(s) of $\mathbf{x}=(x_1,\dots,x_n)$ such that the social welfare, $\sum_i U_i(x_i)$, is maximized, subject to the constraints $x_i\ge 0$ for $1\le i\le n$ and $\sum_i x_i\le C$. In addition, show that if the functions U_i are strictly concave then the allocation maximizing the social welfare is unique.

   (b) Find an explicit expression for the allocation $\mathbf{x}=(x_1,\dots,x_n)$ that maximizes the social welfare, in case $U_i(x_i)=w_i\ln(x_i)$, where for each i, w_i is a given positive weight.

   (c) Show that there exists a unique Nash equilibrium, and characterize it. (Hint: A necessary condition for a Nash equilibrium is $\frac{\partial\pi_i(b)}{\partial b_i}=0$ for $1\le i\le n$. Show that this condition is equivalent to KKT conditions for the solution of maximizing a strictly concave function, as in part (a). Define new valuation functions $\widetilde{U}_i$ by $\widetilde{U}_i'(x_i)=U'(x_i)(1-\frac{x_i}{C})$ for $0<x_i\le C$.)

3. **[Nash saddle point]**
Consider a two person zero sum game represented by a finite $m \times n$ matrix A. The first player selects a probability vector p and the second selects a probability vector q. The first player wishes to minimize pAq^T and the second player wishes to maximize pAq^T. Let V denote the value of the game, so $V = \min_p \max_q pAq^T = \max_q \min_p pAq^T$.

   (a) Consider the following statement S: If $\overline{p}$ and $\overline{q}$ are probability distributions (of the appropriate dimensions) such that $\overline{p}A\overline{q}^T = V$, then $(\overline{p}, \overline{q})$ is a Nash equilibrium point (in mixed strategies). Either prove that statement S is true, or give a counter example.

   (b) Consider the following statement T: A Nash equilibrium consisting of a pair of pure strategies exits if and only if $\min_i \max_j A_{i,j} = \max_j \min_i A_{i,j}$. Either prove that statement T is true, or give a counter example.

4. **[Equilibria for a two player game]**

Consider the two player game shown:

	L	R
T	6,6	2,8
B	8,2	0,0

The first player selects T or B and the second player selects L or R.

   (a) Identify all pure strategy Nash equilibria (if any) and the payoff vector for each one.

   (b) Identify all non-degenerate mixed strategy Nash equilibria and the payoff vector for each one.

   (c) Identify the polytope of all correlated equilibria by giving the set of inequalities they satisfy, and find the correlated equilibria with largest sum of payoffs.