University of Illinois Spring 2013

ECE 586: Exam II

Wednesday, May 1, 2013
7:00 p.m. — 8:30 p.m.
168 Everitt Laboratory

1. **[20 points]** (One-step deviations in a repeated game) Consider a two player repeated game such that the strategy space of the stage game is $\{x, y\}$ for both players. An action sequence for the repeated game is denoted by $a = ((a_1(t), a_2(t)) : t \geq 0)$ and for $t \geq 0$ the corresponding history at time t is $h_t = ((a_1(s), a_2(s)) : 0 \leq s < t)$. Consider the strategy profile $\mu = (\mu_1, \mu_2)$ defined as follows.

$$\mu_1(t, h_t) = \begin{cases} x & t \in \{0, 1\} \\ a_2(t-2) & t \geq 2 \end{cases} \qquad \mu_2(t, h_t) = \begin{cases} y & t = 0 \\ a_2(t-2) & t \text{ even and } t \geq 1 \\ a_1(t-1) & t \text{ odd} \end{cases}$$

   (a) (6 points) What is the action sequence generated by μ?
   Solution:

t	0	1	2	3	4	5	6	$\cdots$
$a_1(t)$	x	x	y	x	y	y	y	$\cdots$
$a_2(t)$	y	x	y	y	y	y	y	$\cdots$

   (b) (7 points) Suppose μ_1' is obtained by deviating from μ_1 at only the single (time, history) pair $(\bar{t} = 3, \bar{h}_{\bar{t}} = ((x, y), (x, x), (x, y)))$. What is the action sequence generated by (μ_1', μ_2)?
   Solution: Same answer as in part (a) because the history doesn't occur under μ.

   (c) (7 points) Suppose μ_2' is obtained by deviating from μ_2 at only the single (time, history) pair $(\bar{t} = 2, \bar{h}_{\bar{t}} = ((x, y), (x, x)))$. What is the action sequence generated by (μ_1, μ_2')?
   Solution:

t	0	1	2	3	4	5	6	7	8	$\cdots$
$a_1(t)$	x	x	y	x	x	y	x	x	x	$\cdots$
$a_2(t)$	y	x	x	y	x	x	x	x	x	$\cdots$

2. **[20 points]** Recall the example of a coalitional game discussed in class (in O&R book) based on a production economy with one capitalist c and a set W of w workers. The capitalist owns a factory that can produce at rate $f(i)$ if there are i workers in the factory, where f is a concave increasing function with $f(0) = 0$. This can be modeled as a coalitional game $\langle N, v \rangle$ with $N = \{c\} \cup W$, and

$$v(S) = \begin{cases} 0 & c \notin S \\ f(|S \cap W|) & c \in S. \end{cases}$$

   Find the vector of payoffs x for this game given by the Shapley value.

   Solution: Consider the $w+1$ agents of the game appearing in a random order. The marginal value of a worker added is zero unless the worker appears after the capitalist. The probability worker one is the $i+1^{st}$ agent to enter is $\frac{1}{w+1}$, and given worker one is the $i+1^{st}$ agent to enter, the conditional probability the capitalist appeared already is $\frac{i}{w}$. Thus, for $1 \leq i \leq w$, the marginal value of worker one is $f(i) - f(i-1)$ with probability $\frac{i}{w(w+1)}$. Taking the expectation over all orders of arrival, and using symmetry, we have $x_1 = \cdots = x_w = \sum_{i=1}^{w} \frac{i(f(i)-f(i-1))}{w(w+1)}$. If

the capitalist arrives after precisely i workers have arrived, the marginal value of the capitalist is $f(i)$, and that happens with probability $\frac{1}{w+1}$ for $0 \leq i \leq w$. Taking the expectation, we find

$$x_c = \frac{1}{w+1} \sum_{i=0}^{w} f(i).$$

Note: It is clear that the Shapley payoffs of the workers are equal, and the sum of all payoffs is the value of the grand coalition, or $f(w)$. Thus, if either of the above two values is worked out, the other can thus be derived from the fact $x_c + wx_1 = f(w)$. This yields the following alternative expression for the payoffs of the workers:

$$x_1 = \cdots = x_w = \frac{f(w)}{w} - \frac{1}{w(w+1)} \sum_{i=0}^{w} f(i) = \frac{1}{w(w+1)} \sum_{i=0}^{w} [f(w) - f(i)]$$

3. **[25 points]** Consider a production economy with two capitalists, a and b, and a set W of $2w$ workers, for some positive integer w. Each of the capitalists owns a factory and the two factories separately produce the same commodity. Each worker can work in only one factory. Let $f(i)$ denote the production rate of one factory with i workers, for $0 \leq i \leq 2w$, where $f(0) = 0$ and f is a concave, increasing function.

   (a) (10 points) This economy can be modeled as a coalitional game $\langle N, v \rangle$. Specify how the set N and characteristic function v should be defined in this case.
   Solution: $N = \{a, b\} \cup W$. Suppose S is a coalition with i workers. Then

$$v(S) = \begin{cases} 0 & \text{if } |\{a,b\} \cap S| = 0 \\ f(i) & \text{if } |\{a,b\} \cap S| = 1 \\ f(\lfloor \frac{i}{2} \rfloor) + f(\lceil \frac{i}{2} \rceil) & \text{if } |\{a,b\} \cap S| = 2 \end{cases}$$

   (b) (15 points) Identify the core of the game.
   Solution: Suppose $x = (x_a, x_b, x_1, \ldots, x_w)$ is in the core. If S is any coalition consisting of one capitalist and w workers, then $v(S) = v(N)/2$. In particular, if S consists of the lower paid capitalist and the w lowest paid workers, then $v(S) = v(N - S) = v(N)/2$. Thus $x(S) = x(N - S)$, which is possible only if the two capitalists have equal payoffs, and all workers have equal payoffs. That is, $x_a = x_b$ and $x_1 = x_2 = \cdots = x_{2w}$.
   It is also required that $x_1 \leq f(w) - f(w-1)$, or else we have the contradiction:
   $x(N - \{1\}) = v(N) - x_1 < v(N) - f(w) + f(w-1) = v(N - \{1\})$.
   Finally, it is also required that $x_1 \geq f(w+1) - f(w)$, or else we have the contradiction:
   For S with one capitalist and $w+1$ workers, $v(S) = f(w) + [f(w+1) - f(w)] > x(S)$.
   Summarizing, if x is in the core, x must satisfy

$$\begin{aligned} &x_1 = \cdots = x_w, \\ &x_1 \in [f(w+1) - f(w), f(w) - f(w-1)] \\ &x_a = x_b = f(w) - wx_1. \end{aligned}$$

   It is easy to check that all such points satisfying these requirements is in the core. While the core is not empty, it is not very large either. The payoffs of the workers are limited to what would typically be a narrow range, in contrast to the case of a single capitalist.

4. **[35 points]** (Revenue optimal sequential selling mechanism) Suppose there is a population of n bidders, and one seller with an object to sell. The value of the object to bidder i is X_i, where $X_1, \ldots, X_n$ are independent, and each is exponentially distributed with parameter

λ. Each bidder i knows his own value, and knows the probability distribution of the other values. The seller knows the probability distribution of all values.

This problem addresses the Bayes-Nash implementation of a revenue optimal seller mechanism, subject to the following protocol. The bidders appear before the seller one at a time in the fixed order $1, 2, \ldots, n$. When a bidder appears, he submits a bid to the seller, and, on the basis of the bid, the seller must either accept or reject the bid. If the bid is accepted, the bidder wins the object and makes a payment. If the bid is rejected, the bidder has no further chance to win and the seller moves on to the next bidder. The mechanism is also constrained to be individually rational. The seller has the option of not selling the object to any bidder, in which case the value of the object to the seller is zero.

(a) (5 points) Briefly describe what the revelation principle implies in the context of this game, and why it applies.

Solution: It means we can take the space of bids of a bidder to be the same as the space of the values, and we can assume without loss of optimality that the Bayes-Nash equilibrium to be produced is the one such that bidders submit their true values as bids. The standard proof of the revelation principle holds under the constraints of this game. That is, given some other mechanism and a Bayes-Nash equilibrium strategy profile $(\beta_i(\cdot))$, the mechanism can be replaced by the direct mechanism such that the direct mechanism first applies the function β_i to the bid of bidder i and feeds the result into the given mechanism.

(b) (5 points) Calculate the virtual valuation function $\psi(x_i) = x_i - \frac{1-F_i(x_i)}{f_i(x_i)}$ of a buyer i. (By symmetry, answer does not depend on i.)

Solution: This was done in a homework problem. $\psi_i(x_i) = x_i - \frac{\text{‘}}{\lambda_i}$.

(c) (10 points) Identify the optimal mechanism in this context for $n = 1$, and calculate the resulting expected revenue R_1^*.

Solution: Seller sells to buyer 1 if and only if $\psi(X_1) \geq 0$, or equivalently, $X_1 \geq \frac{1}{\lambda}$, and the selling price is the minimum the bidder would need to win, namely $\frac{1}{\lambda}$. Therefore, $R_1^* = P\left\{X_1 \geq \frac{1}{\lambda}\right\} \frac{1}{\lambda} = \frac{e^{-1}}{\lambda}$.

(d) (15 points) Let R_n^* denote the maximum expected revenue for the game with n bidders. Describe the actions of the seller for the first bidder in terms of R_{n-1}^*, and find a recursion expressing R_n^* in terms of R_{n-1}^*. *(You can write your answer to this part on the next page.)*

Solution: The seller can either accept or reject the bid of the first bidder, and if the seller rejects the bid, the remaining part of the game has the same form but with $n-1$ bidders and expected revenue R_{n-1}^*. Thus, the payoff of the game can be expressed as

$$E\left[\psi(X_1) I_{\{\text{bidder 1 wins}\}} + R_{n-1}^* I_{\{\text{bidder 1 loses}\}}\right]$$

or, using the fact $\psi(x_i) = x_i - \frac{1}{\lambda}$,

$$R_{n-1}^* + E\left[\left(X_i - \frac{1}{\lambda} - R_{n-1}^*\right) I_{\{\text{bidder 1 wins}\}}\right],$$

under the condition that the conditional probability that bidder 1 wins is a nondecreasing function of his bid. This suggests that the revenue optimal strategy is to accept the bid X_1 of the first bidder if $X_1 \geq \frac{1}{\lambda} + R_{n-1}^*$, and reject it otherwise. This rule satisfies the required monotonicity condition on the conditional probability of bidder 1 winning. Furthermore, we can use the payment rule such that bidder 1 pays $\frac{1}{\lambda_i} + R_{n-1}^*$ if he wins, and zero otherwise.

This results in the recursion

$$\begin{aligned} R_n^* &= R_{n-1}^* + E\left[\left(X_1 - \frac{1}{\lambda} - R_{n-1}^*\right)_+\right] \\ &= R_{n-1}^* + \frac{e^{-(1+\lambda R_{n-1}^*)}}{\lambda} \end{aligned}$$

The recursion is valid for $n = 1$ too if we let $R_0^* = 0$. The recursion can be derived another way. The probability bidder 1 wins, and thus pays $R_{n-1}^* + \frac{1}{\lambda}$, is $P\left\{X_1 \geq \frac{1}{\lambda} + R_{n-1}^*\right\} = e^{-(1+\lambda R_{n-1}^*)}$, and if bidder 1 does not win the expected revenue to seller is R_{n-1}^*.