University of Illinois Spring 2013

ECE 586: Exam I

Monday April 1, 2013
7:00 p.m. — 8:30 p.m.
168 Everitt Laboratory

1. **[30 points]** Recall the standard Cournot game for constants $a > c > 0$. Firm 1 must select a quantity s_1 to produce and firm 2 must select a quantity s_2 to produce, with $s_i \geq 0$. Each firm sells all it produces at the price $a - s_1 - s_2$ per unit, and the nominal production cost is c per unit of production, for either firm.

   Now consider the following variation, which is a game with two stages. Firm 1 has an opportunity to pay an amount $m > 0$ for an equipment upgrade, which would reduce the production cost for firm 1 from c to zero. In the first stage of the game, firm 1 decides whether to buy the equipment upgrade. Firm 2 can observe whether firm 1 buys the equipment upgrade, before the second stage of the game. In the second stage of the game, the firms play the Cournot game to determine s_1 and s_2, with the production cost of firm 1 being either 0 or c, depending on whether firm 1 bought the equipment upgrade. *For simplicity, assume $c \leq \frac{a}{2}$, and no discount factor is used.*

   (a) (10 points) Find the production levels (s_1, s_2) for a subgame perfect equilibrium, and the resulting payoffs of the firms for the second stage of the game, given firm 1 does not buy the equipment upgrade.

   Solution: This is the same as the original Cournot game with payoffs $s_i(a-c-s_1-s_2)$. As found in the homework problems, there is a unique pure strategy NE at $s_1 = s_2 = \frac{a-c}{3}$, with payoff vector $\left(\frac{(a-c)^2}{9}, \frac{(a-c)^2}{9}\right)$.

   (b) (10 points) Find the production levels (s_1, s_2) for a subgame perfect equilibrium, given firm 1 buys the equipment upgrade, and the resulting payoffs of the firms for the second stage of the game. (*Remember we've assumed $c \leq \frac{a}{2}$ for simplicity.*)

   Solution: The payoff functions are $s_1(a - s_1 - s_2)$ and $s_2(a - c - s_1 - s_2)$. Setting the derivative of u_i with respect to s_i equal to zero, we find the best response functions are given by $s_1 = \frac{a-s_2}{2}$ and $s_2 = \frac{a-c-s_1}{2}$, in the range these responses are nonnegative. The fixed point is given by

$$
\begin{aligned}
s_1 &= \frac{a - \frac{a-c-s_1}{2}}{2} = \frac{a+c+s_1}{4} \quad \text{or} \quad s_1 = \frac{a+c}{3} \\
s_2 &= \frac{a - c - \frac{a+c}{3}}{2} = \frac{a-2c}{3}
\end{aligned}
$$

   The NE given by

$$
\left(\frac{a+c}{3}, \frac{a-2c}{3}\right), \text{with payoffs} \left(\frac{(a+c)^2}{9}, \frac{(a-2c)^2}{9}\right)
$$

   (c) (10 points) Identify a subgame perfect equilibrium (SPE) for the overall game, including the decision of firm 1 about the equipment upgrade. In particular, under what conditions on m, a, and c (still under the assumption $c \leq \frac{a}{2}$) does firm 1 buy the equipment upgrade under the SPE?

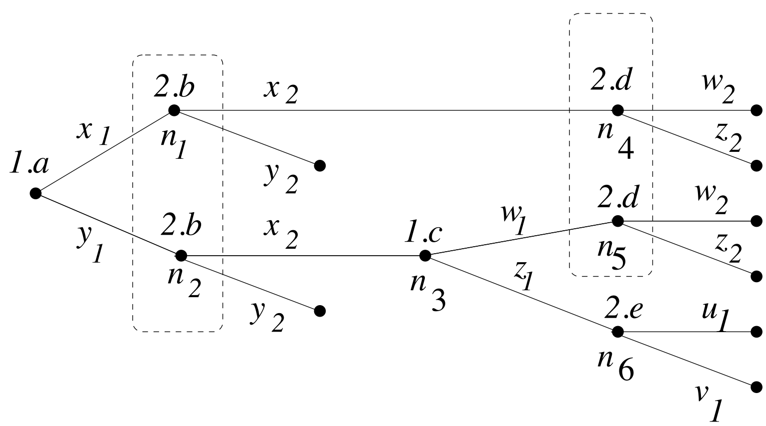
   Solution: If firm 1 buys the equipment upgrade, its profit in the second stage increases from $\frac{(a-c)^2}{9}$ to $\frac{(a+c)^2}{9}$, for an increase of $\frac{4ac}{9}$. Thus, the SPE for the entire game can be described as follows. If $m \le \frac{4ac}{9}$, firm 1 buys the equipment upgrade and subsequently both players follow (b) above, and if $m \ge \frac{4ac}{9}$, firm 1 does not buy the equipment upgrade and both players follow (a) above.

2. **[10 points]** The subparts of this problem are unrelated.

   (a) (5 points) The maxmin value for player i in a finite game with N players is given by $\underline{v}_i = \max_{p_i} \min_{p_{-i}} u_i(p_i, p_{-i})$, where each value of $(p_1, \ldots, p_n)$ represents a mixed strategy profile. If player i uses a Hannan consistent strategy, is it true that with probability one, the limit inferior as $T \to \infty$ of his average payoff per play up to time T will be greater than or equal to $\underline{v}_i$? Briefly explain why.

   Solution: Yes. Hannan consistency means the payoff of player 1 is guaranteed to be at least as large (with probability one, when payoffs are averaged over time, and the time horizon $T \to \infty$) as the payoff for any pure strategy, and hence, as well as the payoff of any time-constant mixed strategy p_i.

   (b) (5 points) In the following extensive game with imperfect information, which one(s) of the six nodes, $n_1, \ldots, n_6$ are root nodes of proper subgames?



   Solution: Only node n_6.

3. **[30 points]** Consider the two-player strategic game with the following payoff matrix, where player 1 selects a row and player 2 selects a column:

	1	2	3
1	6,6	3,4	3,3
2	5,3	9,8	9,7
3	4,3	8,8	8,7

   (a) (5 points) Identify all dominant strategies by either player and state whether each is strongly or weakly dominant.

   Solution: Neither player has a dominant strategy.

   (b) (5 points) Identify all pairs of strategies such that both strategies are for the same player, and one strategy dominates the other, and state whether the domination is strong or weak.

   Solution: For player 1, 2 strongly dominates 3. For player 2, 2 strongly dominates 3.

   (c) (5 points) Identify the pure strategy NE.

   Solution: There are two NE in pure strategies: (1,1) and (2,2).

   (d) (5 points) Identify the NE involving nondegenerate mixed strategies, if any.

   Solution: Any NE must assign zero probability to strictly dominated strategies, so we seek an NE of the form $((a, 1-a, 0), (b, 1-b, 0))$ with $0 < a < 1$ and $0 < b < 1$. Both 1 and 2 should be a best response for player 1, requiring $6b + 3(1-b) = 5b + 9(1-b)$ or $b = 6/7$.

Both 1 and 2 should be a best response for player 2, requiring $6a+3(1-a) = 4a+8(1-a)$, or $a = 5/7$. (There is no NE in which only one of the strategies is a nondegenerate mixed strategy, because the responses to strategies 1 and 2 are unique.) Therefore, $(5/7, 2/7.0), (6/7, 1/7.0)$ is the unique NE including a nondegenerate mixed strategy.

(e) (5 points) Is the game a potential game? If so, identify the potential function. If not, show why.

Solution: To find a candidate potential function, we begin with $\Phi(1,3) = 0$. Then considering deviations of player 2 we find the top row of Φ should be $(3,1,0)$. Then considering deviations of player 1 we find

$$\Phi = \begin{pmatrix} 3,1,0 \\ 2,7,6 \\ 1,6,5 \end{pmatrix}$$

It is readily checked that this Φ is a potential function for the game.

(f) (5 points) Consider the following algorithm for seeking a pure strategy NE. Initially, player 1 selects one of the three pure strategies, all three having equal probability. Then player 2 selects a best response to the strategy of player 1. Then player 1 selects a best response to the strategy of player 2. The process continues. With what probability does the algorithm converge (i.e. eventually each player keeps using the same strategy? Given the algorithm converges, what is the distribution of the limiting strategy pair?

Solution: If player 1 initially selects strategy 1, then player 2 selects strategy 1, and the pair of strategies is $(1,1)$ after all subsequent iterations. If player 1 initially selects strategy 2 or 3, then player 2 selects strategy 2, then player 1 selects 2, and the pair of strategies is $(2,2)$ after all subsequent iterations. Thus, the sequence produced converges with probability one. The limit is $(1,1)$ with probability $\frac{1}{3}$ and $(2,2)$ with probability $\frac{2}{3}$.

4. **[30 points]** Consider the following symmetric, two-player game:

	1	2
1	2,2	1,1
2	1,1	3,3

.

(a) (5 points) Does either player have a (weakly or strongly) dominant strategy?

Solution: No.

(b) (5 points) Identify all the pure strategy and mixed strategy Nash equilibria.

Solution: (1,1) and (2,2) are pure strategy NE. As usual, there is no NE in which only one strategy is a nondegenerate mixed strategy. Seeking a NE of mixed strategies, we try $((a, 1-a), (b, 1-b))$. It is required that $2a+1-a = a+3(1-a)$ or $a = \frac{2}{3}$. Similarly, $b = \frac{2}{3}$. So $\left((\frac{2}{3}, \frac{1}{3}), (\frac{2}{3}, \frac{1}{3})\right)$ is the unique NE using a nondegenerate mixed strategy.

(c) (5 points) Identify all evolutionarily stable pure strategies and all evolutionarily stable mixed strategies.

Solution: It suffices to check the NEs identified in part (b). For either of the two pure strategy NEs, each player uses the unique best response to the play of the other player, so both of those NEs, (1,1) and (2,2), are ESS. It remains to see whether the mixed strategy NE (p,p) given by $p = \left(\frac{2}{3}, \frac{1}{3}\right)$ is an ESS. Since $u(p', p) = u(p, p)$ for all choices of p', the question comes down to whether $u(p, p') > u(p', p')$ for all $p' \neq p$. If $p'(1) > \frac{2}{3}$ then 2 is strictly better than 1 as a response to p', so $u(p, p') < u(p', p')$. This inequality is in the opposite direction needed for ESS, so the mixed NE is not an ESS.

(d) (5 points) The replicator dynamics based on this game represents a large population consisting of type 1 and type 2 individuals. Show that the evolution of the population share vector $\theta(t)$ under the replicator dynamics for this model reduces to a one dimensional ordinary differential equation for $\theta_t(1)$, the fraction of the population that is type 1, and find the differential equation

Solution: The replicator dynamics for the population share vector θ_t are given by $\dot{\theta}_t(a) = \theta_t(a)(u(a, \theta_t) - u(\theta_t, \theta_t))$ for $a \in \{1, 2\}$. Although there are two equations, this system is actually one dimensional because $\theta_t(2) = 1 - \theta_t(1)$. Let $x_t = \theta_t(1)$. Then $u(1, \theta_t) = 2x_t + (1 - x_t) = 1 + x_t$, $u(2, \theta_t) = x_t + 3(1 - x_t) = 3 - 2x_t$, and $u(\theta_t, \theta_t) = x_t(1 + x_t) + (1 - x_t)(3 - 2x_t) = 3 - 4x_t + 3x_t^2$. So the replicator dynamics become $\dot{x}_t = x_t(1 + x_t - 3 + 4x_t - 3x_t^2)$, or, equivalently,

$$\dot{x}_t = x_t(3x_t - 2)(1 - x_t). \tag{1}$$

(e) (5 points) Identify the steady states of the replicator dynamics.

Solution: The righthand side of (1) is zero for $x_t \in \left\{0, \frac{2}{3}, 1\right\}$, so there are three steady states for the replicator dynamics: (1,0), $\left(\frac{2}{3}, \frac{1}{3}\right)$, and (1,0). (NEs are generally steady states of the replicator dynamics, though the converse isn't true.)

(f) (5 points) Of the steady states identified in the previous part, which are asymptotically stable states of the replicator dynamics? Justify your answer.

Solution: The two NE in pure strategies are asymptotically stable (it follows from them being ESS). The mixed strategy NE is not asymptotically stable. From any initial state with $x_0 > \frac{2}{3}$, $x_t \to 1$, and from any initial state with $x_t < \frac{2}{3}$, $x_t \to 0$.