51. Consider the fixed-point equation \( x = Qx + y \), which we would like to solve for \( x \in X \), where \( y \) is an arbitrary vector in \( X \). Since \( Q \) has norm less than 1, the mapping \( T(x) = Qx + y \) is a contraction, and by the Banach contraction mapping theorem, there exists a unique fixed point of \( T \), which can be obtained using successive approximation:

\[
x_o = \lim_{n \to \infty} T^n(\theta) = \sum_{n=0}^{\infty} Q^n y
\]

Hence, there exists a unique \( x_o \) for every \( y \in X \), which implies that there is an operator, say \( A : X \to X \), such that

\[
(I - Q)x = y \quad \Rightarrow \quad x = A(y)
\]

But, by Proposition 1 (p. 147), \( A \) is linear, and is the inverse of \( (I - Q) \). Hence,

\[
[(I - Q)^{-1} - \sum_{n=0}^{\infty} Q^n] y = 0 \quad \forall y \in X
\]

from which the desired result follows.

52. Let \( X_1 \) be the space \( X \) endowed with the norm \( \| \cdot \|_1 \), and \( X_2 \) be the space \( X \) endowed with the norm \( \| \cdot \|_2 \). Let \( A \) be the identity map from \( X_2 \) to \( X_1 \). Clearly, \( A \) is linear, one-to-one, and onto, and hence has a unique inverse which is linear. Furthermore, using the given hypothesis that \( \|x\|_1 \leq \alpha \|x\|_2 \), we obtain

\[
\|A\| = \sup_{x: \|x\|_2 \leq 1} \|Ax\|_1 = \sup_{x: \|x\|_2 \leq 1} \|x\|_1 \leq \alpha \sup_{x: \|x\|_2 \leq 1} \|x\|_2 = \alpha,
\]

and hence that \( A \) is bounded in norm by \( \alpha \). Then, by the Banach inverse theorem (Luenberger, p. 149), the operator inverse of \( A, A^{-1} \), is bounded, whose norm we denote by \( \beta \). In view of this,

\[
\|x\|_2 = \|A^{-1}x\|_2 \leq \beta \|x\|_1 \quad \forall x \in X.
\]

53. We want to find \( A^* : Y^* \to X^* \), defined through the relationship \((A^*y^*)(x) = y^*(Ax)\), where \( y^* \in Y^* = L^*_p = L_q \). Denoting a generic element of \( L_q \), corresponding to \( y^* \), by \( \xi \), we have

\[
y^*(Ax) = \int_0^2 dt \xi(t) \int_0^t K(t, s)x(s) ds \equiv \int_0^2 ds x(s) \int_0^t \xi(t)K(t, s) dt = (A^*y^*)(x)
\]

It then readily follows that

\[
A^* y^* = (A^*\xi)(s) = \int_0^2 K(t, s) \xi(t) dt,
\]
which is an element of $X^* \equiv L_q$. The condition
\[ \int_0^2 \int_0^2 |K(t, s)|^q \, dt \, ds < \infty \]
implies that $A^* \in B(Y^*, X^*)$, just the same way it implies $A \in B(X, Y)$ (see Problem 2, p. 166 of Luenberger).

54.

\[ (A\gamma)(\eta) = \int_{-\infty}^{\infty} \gamma(\xi) \frac{f_{x,y}(\xi, \eta)}{f_2(\eta)} \, d\xi \]

i) \[
A[\alpha \gamma_1 + \beta \gamma_2](\eta) = \int_{-\infty}^{\infty} \left[ \alpha \gamma_1(\xi) + \beta \gamma_2(\xi) \right] \frac{f_{x,y}(\xi, \eta)}{f_2(\eta)} \, d\xi \\
= \alpha \int_{-\infty}^{\infty} \gamma_1(\xi) \frac{f_{x,y}(\xi, \eta)}{f_2(\eta)} \, d\xi + \beta \int_{-\infty}^{\infty} \gamma_2(\xi) \frac{f_{x,y}(\xi, \eta)}{f_2(\eta)} \, d\xi \\
= \alpha (A\gamma_2)(\eta) + \beta (A\gamma_2)(\eta) \\
\Rightarrow \quad \text{linear.}
\]

To show boundedness, let us obtain a bound on $\|A\gamma\|$, where
\[
\|A\gamma\|^2 = \int_{-\infty}^{\infty} d\eta \, f_2(\eta) \left[ \int_{-\infty}^{\infty} \gamma(\xi) \frac{f_{x,y}(\xi, \eta)}{f_2(\eta)} \, d\xi \right]^2.
\]

For each fixed $\eta$,
\[
\left[ \int_{-\infty}^{\infty} \gamma(\xi) f_{x,y}(\xi, \eta) \, d\xi \right]^2 = \left[ \int_{-\infty}^{\infty} \gamma(\xi) \frac{f_{x,y}^{1/2}(\xi, \eta)}{f_2^{1/2}(\eta)} \frac{f_{x,y}^{1/2}(\xi, \eta)}{f_2^{1/2}(\eta)} \, d\xi \right]^2 \\
\leq \int_{-\infty}^{\infty} \gamma^2(\xi) f_{x,y}(\xi, \eta) \, d\xi \cdot \int_{-\infty}^{\infty} f_{x,y}(\xi, \eta) \, d\xi \\
\quad \uparrow \text{Cauchy Schwartz inequality}
\]

Hence, using this bound in (\star), we arrive at:
\[
\|(A\gamma)\|^2 \leq \int_{-\infty}^{\infty} d\eta \, f_2(\eta) \frac{f_2(\eta)}{f_2^2(\eta)} \int_{-\infty}^{\infty} \gamma^2(\xi) f_{x,y}(\xi, \eta) \, d\xi \\
= \int_{-\infty}^{\infty} d\eta \int_{-\infty}^{\infty} \gamma^2(\xi) f_{x,y}(\xi, \eta) \, d\xi = \int_{-\infty}^{\infty} d\xi \, \gamma^2(\xi) \int_{-\infty}^{\infty} \frac{f_{x,y}(\xi, \eta) \, d\eta}{f_2(\xi)} \\
= \|\gamma\|^2.
\]

Hence, $\|(A\gamma)\|^2 \leq \|\gamma\|^2 \Rightarrow$ bounded.
ii) 
\[ \|A\| = \sup_{\|\gamma\|} \|A\gamma\| \leq 1 \] by the bound in (i).

Taking \( \gamma \equiv 1 \), a.e. under Lebesgue measure, we obtain
\[ \|A\| = \|A\| = 1 \]

iii) 
\[ (A\gamma, \beta)_2 = \int_{-\infty}^{\infty} d\eta f_2(\eta) \beta(\eta) \int_{-\infty}^{\infty} f_{x,y}(\xi, \eta) d\xi \]
\[ = \int_{-\infty}^{\infty} d\xi f_1(\xi) \gamma(\xi) \int_{-\infty}^{\infty} \beta(\eta) \frac{f_{x,y}(\xi, \eta)}{f_1(\xi)} d\eta = (\gamma, A^* \beta)_1 \]
\[ \uparrow \text{ interchange order of integration} \]
\[ \therefore A^*(\beta) = \int_{-\infty}^{\infty} \beta(\eta) \frac{f_{x,y}(\xi, \eta)}{f_1(\xi)} d\eta = E[\beta(y)|x] \]
\[ \uparrow \text{ conditional expectation.} \]

55. i) The first statement is true, that is the norm of \( A^*A \) is equal to square of the norm of \( A \). To prove this result, first note that
\[ \|A^*A\| \leq \|A^*\| \|A\| = \|A\|^2 \]
\[ \uparrow \text{ Thm 1, p. 151} \]

while
\[ \|A\|^2 = \sup_{\|x\|=1} (Ax, Ax) = \sup_{\|x\|=1} (x, A^*Ax) \leq \sup_{\|x\|=1} \|A^*A\| = \|A^*A\| \]
\[ \uparrow \text{ (*) and (**) now verify the desired result.} \]

Note that if we had started with \( AA^* \) (instead of \( A^*A \)) following the same steps as above would have led to \( \|AA^*\| = \|A\|^2 \).

ii) The second statement is also true, and that follows readily from the last result above, since the norms of both \( AA^* \) and \( A^*A \) are equal to \( \|A\|^2 \) (which itself is equal to \( \|A^*\|^2 \)).

56. The pseudo-inverse can be obtained in two ways. The first one uses range and null space arguments (given immediately below), and the second one uses a formula given in class (see
also Problem 19 on p. 167 of the text).

\[
A = \begin{pmatrix}
1 & 2 & 1 \\
1 & 1 & 0 \\
1 & 1 & 0
\end{pmatrix}; \quad \mathcal{N}(A) = \text{span} \begin{pmatrix}
1 \\
-1 \\
1
\end{pmatrix}
\]

\[\mathcal{N}(A)^\perp = \text{span} \left\{ \begin{pmatrix}
1 \\
1 \\
0
\end{pmatrix}, \begin{pmatrix}
1 \\
0 \\
-1
\end{pmatrix} \right\}.\]

In view of the above, let \( x_1 := (1 1 0)', \ x_2 := (1 0 -1)', \) both of which are in \( \mathcal{N}(A)^\perp \). Further let

\[y_1 = Ax_1 = \begin{pmatrix} 3 \\ 2 \\ 2 \end{pmatrix}, \quad y_2 = Ax_2 = \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}\]

Then,

\[\mathcal{R}(A) = \text{span} \{y_1, y_2\}; \quad \mathcal{R}(A)^\perp = \text{span} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\]

Since \( A^\dagger \) maps \( y_1 \) to \( x_1 \), \( y_2 \) to \( x_2 \), and every element in \( \mathcal{R}(A)^\perp \) to \( \theta \), it follows that,

\[A^\dagger = \begin{bmatrix}
0 & 1 & 1 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{bmatrix}\begin{bmatrix}
0 & 0 & 3 \\
-1 & 1 & 2 \\
1 & 1 & 2
\end{bmatrix}^{-1} = \frac{1}{6} \begin{bmatrix}
-2 & 3 & 3 \\
2 & 0 & 0 \\
4 & -3 & -3
\end{bmatrix}\]

For the second derivation, use the formula :

\[A^\dagger = \lim_{\epsilon \to 0^+} A^* (AA^* + \epsilon I)^{-1}\]

Simple computation yields:

\[A^* (AA^* + \epsilon I)^{-1} = \frac{1}{\epsilon^2 + 10\epsilon + 6} \begin{bmatrix}
\epsilon - 2 & \epsilon + 3 & \epsilon + 3 \\
2\epsilon + 2 & \epsilon & \epsilon \\
\epsilon + 4 & -3 & -3
\end{bmatrix}\]

from which the expression for \( A^\dagger \) given above follows by letting \( \epsilon \to 0^+ \).