Solution Set 1

1. Let \( x \in M + N \). Then \( x = m + n \) with \( m \in M, n \in N \). Clearly \( m \in M \cup N, n \in M \cup N \) and \( x = m + n \in [M \cup N] \). Thus \( M + N \subseteq [M \cup N] \).

Let \( x \in [M \cup N] \). Then \( x = \sum_{i=1}^{k} d_i y_i \) with \( y_i \in M \) or \( y_i \in N \) (or both). Thus \( x = m + n \), where

\[
m = \sum_{i\in M \cup N} d_i y_i \in M
\]

\[
n = \sum_{i\in N} d_i y_i \in N.
\]

Thus \( [M \cup N] \subseteq M + N \), and combined with above we have equality.

2. We must show that (i) \( K \subseteq \text{co} (S) \)
   (ii) \( K \supset \text{co} (S) \)

(i) \( K \subseteq \text{co} (S) \)

\[k \in K \Rightarrow k = \sum_{i=1}^{m} \alpha_i x_i, \text{ where } \sum_{i=1}^{m} \alpha_i = 1, \quad \alpha_i > 0, x_i \in S.\]

Now

\[x_i \in S \Rightarrow x_i \in \text{co} (S).\]

\[\text{co} (S) \text{ convex } \Rightarrow \frac{\alpha_1 x_1}{\alpha_1 + \alpha_2} + \frac{\alpha_2 x_2}{\alpha_1 + \alpha_2} = y_1 \in \text{co} (S).\]

Similarly,

\[\frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2 + \alpha_3} y_1 + \frac{\alpha_3 x_3}{\alpha_1 + \alpha_2 + \alpha_3} = y_2 \in \text{co} (S).\]

But

\[y_2 = \frac{\alpha_1 x_1}{\alpha_1 + \alpha_2 + \alpha_3} + \frac{\alpha_2 x_2}{\alpha_1 + \alpha_2 + \alpha_3} + \frac{\alpha_3 x_3}{\alpha_1 + \alpha_2 + \alpha_3}.\]

Continuing in this manner, and noting that \( \sum_{i=1}^{n} \alpha_i = 1 \),

\[y_{n-1} = \frac{\alpha_1 x_1}{\sum_{i=1}^{n} \alpha_i} + \cdots + \frac{\alpha_n x_n}{\sum_{i=1}^{n} \alpha_i} = \sum_{i=1}^{n} \alpha_i x_i \in \text{co} (S).\]

Therefore,

\[K \subseteq \text{co} (S).\]
(ii) $\text{co} (S) \subset K$

$$\forall x, y \in K \exists \alpha_i, x_i, \alpha'_i, x'_i : x = \sum_{i=1}^{n} \alpha_i x_i$$

and

$$y = \sum_{i=1}^{n} \alpha'_i x'_i$$

where

$$\sum_{i=1}^{n} \alpha_i = \sum_{i=1}^{n} \alpha'_i = 1.$$  

Now, for $0 < \lambda < 1$,

$$\lambda y + (1 - \lambda) x = \lambda \sum_{i=1}^{n} \alpha'_i x'_i + (1 - \lambda) \sum_{i=1}^{n} \alpha_i x_i$$

where

$$\lambda \sum_{i=1}^{n} \alpha'_i + (1 - \lambda) \sum_{i=1}^{n} \alpha_i = 1.$$  

$\implies \lambda y + (1 - \lambda) x \in K$. Therefore, $K$ is convex. Obviously $S \subset K$, and hence $\text{co} (S) \subset K$ by definition.

3. By Hölder’s inequality, for $x \in L_p[-1, 1]$ and $y \in L_q[-1, 1]$, with $(1/p) + (1/q) = 1$, we have

$$\int_{-1}^{1} |x(t)y(t)| dt \leq \|x\|_p \|y\|_q$$

with equality if, and only if,

$$|x(t)| = \left(\frac{|y(t)|}{\|y\|_q}\right)^\frac{p}{q} \|x\|_p$$

Now, for the problem at hand, take $y(t) = t^3$ and $p = 3$, which makes $q = 3/2$. Furthermore, $\|x\|_p = 2^{\frac{3}{2}}$. Hence, from Hölder’s inequality,

$$\int_{-1}^{1} |x(t)t^3| dt \leq 2^{\frac{3}{2}} \|t^3\|_{\frac{2}{3}} = (2)^{\frac{3}{2}} (11)^{-\frac{3}{2}},$$

which says that $\int_{-1}^{1} |x(t)t^3| dt$ cannot be larger than $(2)^{\frac{3}{2}} (11)^{-\frac{3}{2}}$, and in fact this upper bound can be achieved (from the equality above) by choosing

$$|x(t)| = \left(t^3(2)^{-\frac{3}{2}} (11)^{\frac{3}{2}}\right)^{\frac{1}{3}} 2^{\frac{1}{3}} = (2)^{-\frac{1}{3}} (11)^{\frac{1}{3}} |t|^\frac{3}{2}$$
Now,
\[ \int_{-1}^{1} t^3 x(t) dt \leq \int_{-1}^{1} |t^3 x(t)| dt \]
where we would have equality if \( t^3 x(t) \) is picked so that \( t^3 x(t) \) is equal to \( |t^3 x(t)| \), that is the sign of \( x(t) \) should be the same as the sign of \( t^3 \), and hence of \( t \). Hence, the unique solution to the maximization problem is
\[ x(t) = (2)^{-\frac{1}{3}} (11)^{\frac{1}{3}} |t|^\frac{3}{2} \text{ sgn} (t) \]

4. Noting the obvious relationship,
\[ \min \int_{-1}^{1} t^3 x(t) dt \geq \min \left( - \int_{-1}^{1} |t^3 x(t)| dt \right) = - \max \int_{-1}^{1} |t^3 x(t)| dt \]
it readily follows from the solution to Problem 3 above that the following choice of \( x(\cdot) \) makes the inequality above an equality, and hence constitutes a solution (and a unique one) to the minimization problem:
\[ x(t) = -(2)^{-\frac{1}{3}} (11)^{\frac{1}{3}} |t|^\frac{3}{2} \text{ sgn} (t) \]

5. Consider, instead of \( f \), the functional
\[ g(x) = \int_{0}^{3} |x(t) \cos \pi t| dt . \]
By Hölder’s inequality,
\[ g(x) \leq \|x\|_2 \| \cos \pi t\|_2 \leq \| \cos \pi t\|_2 = \frac{1}{\sqrt{2}} \left( \int_{0}^{3} (1 + \cos 2\pi t) dt \right)^{\frac{1}{2}} = \sqrt{3}/2 , \]
and equality holds, i.e., \( g(x) = \sqrt{3}/2 \), if and only if
\[ |x(t)| = \frac{|\cos \pi t|}{\| \cos \pi t\|_2} = \left( \sqrt{2}/3 \right) |\cos \pi t| . \]
Since \( f(x) \geq -g(x) \geq -\sqrt{3}/2 \), if we can find an \( x \), with \( \|x\|_2 = 1 \), such that \( f(x) = -\sqrt{3}/2 \), then this would be a minimizing solution. Now pick
\[ x^o(t) = -\left( \sqrt{2}/3 \right) \cos \pi t \quad \Rightarrow \quad f(x^o) = -\sqrt{3}/2 \]
Hence, \( x^o \) is a minimizing solution. From Hölder’s inequality, every maximizing solution for \( g \) has to satisfy the equality
\[ |x(t)| = \frac{|\cos \pi t|}{\| \cos \pi t\|_2} \]
and by inspection \( x^o \) is the only one that minimizes \( f \). Hence the solution is unique.
6. This result basically follows from the application of Hölder’s inequality and conditions under which this inequality is actually an equality. Assume, without any loss of generality, that \( x \neq \theta \). First note the inequality:
\[
\int_0^1 |x(t) + y(t)|^p dt \leq \int_0^1 |x(t) + y(t)|^{p-1}|x(t)| dt + \int_0^1 |x(t) + y(t)|^{p-1}|y(t)| dt \quad (\ast)
\]
with equality holding if \( x(t) + y(t) = \max\{|x(t)|, |y(t)|\} |x(t)| + |y(t)| \quad \forall t \in [0, 1] \).

Note that equality holds in (\ast) iff
\[
|x(t) + y(t)| = |x(t)| + |y(t)| \quad \forall t \in [0, 1],
\]
and this can only happen if \( y(t) \) and \( x(t) \) have the same sign for each \( t \).

Now, since \( x + y \in L[0, 1] \), from Hölder’s inequality (applied to each term in (\ast)),
\[
\int_0^1 |x(t) + y(t)|^p dt \leq \left( \int_0^1 |x(t) + y(t)|^{(p-1)q} dt \right)^\frac{1}{p} \left[ \|x\|_p + \|y\|_p \right]
\]
and dividing both sides by \( \left( \int_0^1 |x(t) + y(t)|^p dt \right)^\frac{1}{p} \), and using \( 1 - \frac{1}{q} = \frac{1}{p} \), we obtain
\[
\|x + y\|_p \leq \|x\|_p + \|y\|_p
\]
with equality holding (from Hölder’s inequality) iff
\[
\left\{ \frac{|x(t) + y(t)|^{p-1}}{\|x + y\|^{p-1}_q} \right\} = \left( \frac{|x(t)|}{\|x\|_p} \right)^{\frac{1}{q}} \left( \frac{|y(t)|}{\|y\|_p} \right)^{\frac{1}{q}} \quad \forall t \in [0, 1],
\]
in addition to (\ast). Raising the above to \( pq \)-th power, we obtain
\[
\frac{|x(t) + y(t)|^{q+p-q}}{\|x + y\|^{p-1}_q} = \frac{|x(t)|^p}{\|x\|_p^p} = \frac{|y(t)|^p}{\|y\|_p^p}
\]
\[ \Rightarrow \]
\[
|x(t)| = \{\|x\|_p/\|y\|_p\} \quad \forall t
\]
\[ \Leftrightarrow \] holds iff, for some positive constant \( \alpha \), \( |x(t)| = \alpha |y(t)| \), or \( y(t) = \theta \); in the latter case
\[
\frac{|x(t) + y(t)|^p}{\|x + y\|^{p-1}_q} = \frac{|x(t)|^p}{\|x\|_p^p}, \quad \text{with} \ x \neq \theta .
\]

In view of the condition (\ast\ast), the former holds iff \( x(t) = \alpha y(t) \) \quad \forall t.

This completes the proof of the desired result, that \( L_p(0, 1) \) is strictly (strongly) normed.

7. As hinted, this is a proof by contradiction. Let \( Z \) be the subspace spanned by \( \{x_1, \ldots, x_n\} \).

Note that by a property of the norm (which follows from the triangle inequality, as discussed in class), \( \|y - z\| \geq \|z\| - \|y\| \) for any \( z \in Z \). Further,
\[
\min_{z \in Z} \|y - z\| \leq \|y\|
\]
which is obtained by simply taking $z = \theta \in Z$. Hence, if $z_0$ is a minimizing solution,

$$\|z_0\| - \|y\| \leq \|y - z_0\| \leq \|y\| \Rightarrow \|z_0\| \leq 2\|y\|,$$

which says that the solution belongs to a subset of $Z$ whose elements satisfy the norm bound $\|z\| \leq 2\|y\|$. (Note that $\|y\|$ is fixed and finite.) Call this subset $\tilde{Z}$. Let $z_1 \in \tilde{Z}$ and $z_2 \in \tilde{Z}$ be two solutions to the minimization problem. Then, by definition,

$$\|y - z_1\| = \|y - z_2\| \leq \|y - z\| \forall z \in \tilde{Z}. \quad (i)$$

Now let $z = (z_1 + z_2)/2$, which also belongs to $\tilde{Z}$. Then, from (i)

$$\|y - z_1\| \leq \left\| y - \frac{z_1 + z_2}{2} \right\|. \quad (ii)$$

However, using the scaling property of the norm, and the triangle inequality:

$$\left\| y - \frac{z_1 + z_2}{2} \right\| = \frac{1}{2} \|y - z_1 + y - z_2\| \leq \frac{1}{2} \{\|y - z_1\| + \|y - z_2\|\} = \|y - z_1\|.$$

Hence $\|y - z_1\|$ is bounded from above and below by the same quantity, leading to

$$\|y - z_1 + y - z_2\| = 2\|y - z_1\| = \|y - z_1\| + \|y - z_2\|.$$  

Since $X$ is strictly normed, this implies that either $y - z_2 = \theta \Rightarrow y = z_2$ a.e. or $y - z_1 = \alpha(y - z_2)$ for some $\alpha$.

The first possibility says that $y \in \tilde{Z}$, in which case there is a unique representation of $y$ in terms of the basis vectors $\{x_1, \ldots, x_n\} \Rightarrow z_1 = z_2$ a.e. Under the second condition, if $\alpha = 0$, again $y \in \tilde{Z} \Rightarrow z_1 = z_2$ a.e. If $\alpha \neq 0$, $(1 - \alpha)y = z_1 - \alpha z_2$ which again says that $y \in \tilde{Z}$ unless $\alpha = 1$. If $\alpha = 1$, however, $y - z_1 = \alpha(y - z_2) \Rightarrow z_2 = z_1$ a.e. Hence, the minimizing solution has to be unique.

8 (a)

$$x \in \ell_{p_0} \Rightarrow \sum_{i=1}^{\infty} |\xi_i|^{p_0} < \infty; \quad 1 \leq p_0 < \infty.$$

$\Leftrightarrow$ Given an $\epsilon > 0$, $\exists N_0(\epsilon) \ni$

$$\left( \sum_{i=N+1}^{\infty} |\xi_i|^{p_0} \right)^{\frac{1}{p_0}} < \epsilon \forall N > N_0(\epsilon).$$

Choosing $\epsilon < 1$, we can readily take $|\xi_i| < 1 \forall i > N_0$. Then, since $p > p_0 \geq 1$,

$$|\xi_i|^{p_0} \geq |\xi_i|^p \forall i > N_0$$

$\Rightarrow$

$$e^{p_0} > \sum_{i=N+1}^{\infty} |\xi_i|^{p_0} \geq \sum_{i=N+1}^{\infty} |\xi_i|^p \forall N > N_0(\epsilon)$$

5
\[
\left\{ \sum_{i=1}^{\infty} |\xi_i|^p \right\}^{\frac{1}{p}} < \epsilon \quad \forall \epsilon > N_0(\epsilon)
\]

\[
\sum_{i=1}^{\infty} |\xi_i|^p = \sum_{i=1}^{N} |\xi_i|^p + \sum_{i=N+1}^{\infty} |\xi_i|^p < \sum_{i=1}^{N} |\xi_i|^p + \epsilon p_0 < \infty.
\]

Hence, \( x \in \ell_{p_0} \Rightarrow x \in \ell_p, \ p > p_0, \ p < \infty. \)

(b) Since we will be taking the limit on \( \|x\|_p \) as \( p \to \infty \), we have to restrict attention to those sequences \( x = \{\xi_1, \xi_2, \ldots, \xi_n, \ldots\} \) which belong to \( \ell_{\bar{p}} \) for some finite \( \bar{p} \geq 1 \). As we know from part (a), \( x \in \ell_{\bar{p}} \Rightarrow x \in \ell_p \ \forall p > \bar{p}, \ p < \infty. \) Furthermore, it again follows from the discussion in part (a) that given a sufficiently small \( \epsilon > 0, \exists \) an integer \( N_0(\epsilon) \) \( \sum_{i=1}^{N} |\xi_i|^p \frac{1}{p} < \epsilon \ \forall \epsilon > N_0(\epsilon). \)

This implies that there exists an \( \bar{n} \) (for any given \( x \in \ell_{\bar{p}} \)) such that \( \sup_{i} \{ |\xi_i| \} = \bar{n}. \)

Let \( \bar{\xi}_i = \xi_i/\bar{n} \); clearly \( |\bar{\xi}_i| \leq 1 \ \forall i = 1, 2, \ldots. \) Now,

\[
\|x\|_p = |\bar{n}| \left\{ \sum_{i=1}^{\infty} |\bar{\xi}_i|^p \right\}^{\frac{1}{p}} \Delta = |\bar{n}| (\eta_p)^{\frac{1}{p}}
\]

where

\[
\eta_p = \sum_{i=1}^{\infty} |\bar{\xi}_i|^p < \infty \quad \forall p \geq \bar{p}.
\]

Since \( |\bar{\xi}_i| \leq 1; \eta_p \leq \eta_{p'} \) for \( p' > p. \) Therefore, \( \{\eta_p\}_{p}^{\infty} \) is a nonincreasing and bounded (from below by zero) sequence \( \Rightarrow \exists K \ni \lim_{p \to \infty} \eta_p = K < \infty. \)

\[
\therefore \lim_{p \to \infty} \|x\|_p = |\xi| \lim_{p \to \infty} \{\eta_p\}^{\frac{1}{p}} = |\xi| \lim_{p \to \infty} K^{\frac{1}{p}} = |\xi|.
\]

Hence,

\[
\|x\|_p = \sup_{i} \{ |\xi_i| \} = \|x\|_\infty.
\]

[Note that \( \lim_{p \to \infty} \{\eta_p\}^{\frac{1}{p}} = \lim_{r \to \infty} \{\lim_{p \to \infty} \eta_p\}^{\frac{1}{p}} \) because \( \eta_p \) is a nonincreasing and bounded sequence.]

9. \( n(x, y) = \|x - y\|/\left[1 + \|x - y\|\right]. \)

(i) \( n(x, y) \geq 0 \) because \( \|\cdot\| \) is a norm, and is \( = 0 \) iff \( x = y \) because \( \|x - y\| = 0 \) iff \( x = y. \)

(ii) \( n(x, y) = n(y, x) \) – trivially true.

(iii) \( n(x, y) \leq n(x, z) + n(y, z) \) – to be shown below.
Let
\[ \|x - y\| = a, \quad \|x - z\| = b, \quad \|y - z\| = c, \quad a, b, c \geq 0 \]
By triangle inequality,
\[ a \leq b + c \leq b + c + 2bc + abc \]
Add \( ab + ac + abc \) to both sides:
\[ a + ab + ac + abc \leq b + c + ab + ac + 2bc + 2abc \quad \Leftrightarrow \quad a(1 + b)(1 + c) \leq (1 + a)(b + 2bc + c) \]
Divide throughout by \( (1 + a)(1 + b)(1 + c) > 0 \):
\[ \frac{a}{1 + a} \leq \frac{b + 2bc + c}{1 + b + c} \]
which is the desired inequality.

10. We have to show that

(i) \( \|\alpha[x]\| = |\alpha|\|x\| \quad \forall \alpha \in \mathbb{R} \)
(ii) \( \|\{x\} + \{y\}\| \leq \|\{x\}\| + \|\{y\}\| \quad \forall \{x, y\} \in X/B \)
(iii) \( \|\{x\}\| \geq 0 \quad \& \quad = 0 \iff \{x\} = 0 \).

(i)
\[ \|\alpha[x]\| = \inf_{m \in M} |\alpha x + m| = |\alpha| \inf_{m' \in M} |x + m'| \quad (\text{where } m' = \frac{m}{\alpha}) \]
\[ = |\alpha||\{x\}||. \]

In the above, the second equality holds because \( M \) is a subspace.

(ii)
\[ \|\{x\} + \{y\}\| = \inf_{m \in M} |x + y + m| = \inf_{m' \in M, m'' \in M} |x + m' + y + m''| \]
\[ \leq |x + m'| + |y + m''| \quad \forall m', m'' \in M. \]
Since this is true for all \( m', m'' \), clearly
\[ \|\{x\} + \{y\}\| \leq \inf_{m' \in M} |x + m'| + \inf_{m'' \in M} |y + m''| = \|\{x\}\| + \|\{y\}\|. \]

(iii) \( \|\{x\}\| \) is nonnegative because
\[ |x + m| \geq 0 \quad \forall m \in M, x \in X. \]
Then, all we need to show is that
\[ \|\{x\}\| = 0 \Rightarrow \{x\} = 0. \]
Since, \( \forall m \in M \) and for any \( x \in X \),
\[ |x + m| \leq |x| + |m| = |x| \quad \text{and} \quad |x + m| \geq |x| - |m| = |x| \]
we have \( |x + m| = |x| \). This implies that
\[ \|\{x\}\| = \inf_{m \in M} |x + m| = |x| \quad \Leftrightarrow \quad x \in M \quad \Leftrightarrow \quad \{x\} = 0 \]
This completes the verification.