I start with a definition – that of a sublinear functional.

**Definition.** Let $X$ be a real vector space (not necessarily normed). A map $p : X \rightarrow \mathbb{R}$ is called a sublinear functional if it satisfies the following two properties:

\[ p(x + y) \leq p(x) + p(y) \quad \forall x, y \in X \quad \text{(subadditive)} \]
\[ p(\alpha x) = \alpha p(x) \quad \forall \alpha \geq 0, \forall x \in X \quad \text{(positive homogeneity)} \]

Note that “norm” is a sublinear functional.

The next result is a useful property of sublinear functionals.

**Lemma.** Let $M \subset X$ be a subspace, and $f$ a linear functional on $M$, such that for some sublinear functional $p,$

\[ f(x) \leq p(x) \quad \forall x \in M. \]

Let $x_o$ be a fixed element of $X$. Then, for any real number $c$, the following are equivalent:

\[ f(x) + \lambda c \leq p(x + \lambda x_o) \quad \forall x \in M, \lambda \in \mathbb{R} \quad (1) \]
\[ -p(-x - x_o) - f(x) \leq c \leq p(x + x_o) - f(x) \quad \forall x \in M \quad (2) \]

Furthermore, there is a real number $c$ satisfying (2) and hence (1).

**Proof:** To show that (1) $\Rightarrow$ (2), first set $\lambda = 1$, and then set $\lambda = -1$ and replace $x$ by $-x$; then (2) is immediate. To show the converse, first take $\lambda > 0$, and replace $x$ by $x/\lambda$ on the RHS inequality, to obtain (1). Now take $\lambda < 0$, and replace $x$ by $x/\lambda$ on the LHS inequality, to arrive again at (1). For $\lambda = 0$, (1) is always satisfied, from the definitions of $f$ and $p$. To produce the desired $c$, let $x, y$ be arbitrary elements out of $M$. Then,

\[ f(x) - f(y) = f(x - y) \leq p(x - y) \]
\[ \leq p(x + x_o) + p(-y - x_o) \quad \text{by subadditivity} \]
\[ \Rightarrow \quad -p(-y - x_o) - f(y) \leq p(x + x_o) - f(x) \quad \forall x, y \in M \]
\[ \Rightarrow \quad \sup_{y \in M} \{-p(-y - x_o) - f(y)\} \leq \inf_{x \in M} \{p(x + x_o) - f(x)\} \]

Any $c$ between the “sup” and “inf” values will do it. $\diamond$

Now we have the main theorem.

**Theorem (Hahn-Banach).** Given a real linear vector space $X$ and a subspace $M$, let $p$ be a sublinear functional on $X$, and $f$ a linear functional on $M$ such that $f(x) \leq p(x) \quad \forall x \in M$. Then, there is a linear functional $F$ on $X$ such that $F = f$ on $M$ and $F \leq p$ on all of $X$.

**Proof.** Let $x_0 \notin M, x_0 \in X$, and consider the subspace $M_0$ consisting of all elements $x + \lambda x_0, x \in M, \lambda \in \mathbb{R}$. We may extend $f$ to a linear functional on $M_0$ by defining

\[ f_0(x + \lambda x_0) = f(x) + \lambda c \]
where $c$ is any real number. Now choose $c$ such that it satisfies condition (2) in the previous Lemma. Then,

$$f_0(x + \lambda x_0) \leq p(x + \lambda x_0), \quad \iff \quad f_0 \leq p \quad \text{on } M_0$$

Hence, we have seen how to extend $f$ from $M$ to $M_0$, which is of dimension one higher than that of $M$ (assuming that $M$ is finite-dimensional).

1. If $X$ is normed and separable, then we can find a countable set of vectors $\{x_0, x_1, \ldots\}$, all linearly independent and not in $M$, so that every element in $X$ can be approximated to any degree by linear combinations of vectors out of $M$ and $\{x_0, x_1, \ldots\}$. Let $M_i = M_{i-1} + \lambda x_i$, which is a subspace. If $y \in M_i$, we can write it as $y = x + \lambda x_i$, where $x \in M_{i-1}$ and $\lambda \in \mathbb{R}$. Define $f_i(x + \lambda x_i) = f_{i-1}(x) + \lambda c_i$

where $c_i$ is as in the Lemma with $f$ replaced by $f_{i-1}$. This then shows how we can extend $f$ recursively to a countable dense subset of $X$. Call this extension $g$, which is naturally linear. By construction, $g \leq p \quad \forall x \in S$, where $S$ is a dense subset of $X$. Now, to extend $g$ to $X$, let $x \in X$ be given and $\{s_n\}$ be a sequence out of $S$ converging to $x$. Then, the limit $\lim_{n \to \infty} g(s_n)$ is well defined; call this $F(x)$. This defines $F$ pointwise. $F$ is obviously linear. Furthermore, since $g(s_n) \leq p(s_n) \quad \forall s_n \in S$, we have $F(x) \leq p(x) \quad \forall x \in X$.

2. If $X$ is not a separable normed space, we cannot find an ordered set of vectors to form a basis for a dense subset. Then, the proof will have to be modified, where the strict ordering is replaced by partial ordering, and Zorn's Lemma is used. Zorn’s Lemma says that: “If $F$ is a partially ordered set in which every chain has an upper bound, then $F$ possesses a maximal element.” This is equivalent to the Axiom of Choice, which says the following: “Given a nonempty class of disjoint nonempty sets, a set can be formed which contains precisely one element taken from each set in the given class.” To use this in our proof, let $C$ be the collection of all pairs $(h, H)$ where $h$ is an extension of $f$ to the subspace $H \supseteq M$, and $h \leq p$ on $H$. Partially order $C$ by $(h_1, H_1) \leq (h_2, H_2)$ iff $H_1 \subset H_2$ and $h_1 \leq h_2$ on $H_1$. Then, every chain in $C$ has an upper bound (consider the union of all subspaces in the chain). This implies by Zorn’s Lemma that $C$ has a maximal element. Call this $(F, F)$. If $F \neq X$, then this means that we can extend $F$ to a larger subspace – but this contradicts its maximality. Hence $F$ is defined on $X$.

The two corollaries below follow from the Hahn-Banach theorem. I provide a proof for only the first one.

Corollary 1. Let $f$ be a continuous linear functional defined on a subspace $M$ of $X$, a normed linear space. Then, there exists $F \in X^*$ such that $\|f\|_M = \|F\|_X$ and $F$ is an extension of $f$.

Proof. Take $p(x) = \|f\|_M \|x\|_X$ in the Hahn-Banach Theorem, which is clearly a sublinear functional. Now note that

$$F(x) \leq p(x) = \|f\|_M \|x\|_X \quad \text{and} \quad -F(x) = F(-x) \leq \|f\|_M \|x\|_X \quad \forall x \in X,$$

which leads to

$$|F(x)| \leq \|f\|_M \|x\|_X \quad \Rightarrow \quad \sup_{x \in X, \|x\| \leq 1} |F(x)| \leq \|f\|_M.$$

But since $\sup_{x \in X, \|x\| \leq 1} |F(x)| \geq \sup_{x \in M, \|x\| \leq 1} |F(x)| = \|f\|_M$, we actually have equality above. Since $F$ is bounded, it is also continuous.

Corollary 2. Let $x_0 \in X$, a normed linear space. Then, there exists a nonzero bounded linear functional $F$ on $X$ such that

$$F(x_0) = \|F\| \|x_0\|.$$

The converse of this last result does not generally hold; that is, given $F \in X^*$, we may not be able to find $x \in X$ such that $F(x) = \|F\| \|x\|$. See, for example, Example 1 on page 113 of the text by Luenberger. We will see (in class) that for some (but not all) Banach spaces this converse indeed holds.