ASSIGNMENT 4

Reading Assignment:    Text: Chapter 4. Correspondence 10
Suggested Reading:    Curtain & Pritchard: Chp 5 (pp. 75-84).
                        Review probability theory and stochastic processes from any (graduate) text of your choice.

Notice :    On February 28, we will start class at 9:30 am

Problems (to be handed in):    Due Date:   Thursday, February 28.

This first problem of this set is related to the topic of “wavelets” (which I briefly introduced in class), but no prior knowledge of wavelets is necessary to solve it.

33. Let \( J \) be an index set, and \( \{\xi_j\}_{j \in J} \) a family of functions in a (complex) Hilbert space \( H \). This family is called a frame if there exist constants \( A > 0, B < \infty \) such that for all \( f \in H \),

\[
A \|f\|^2 \leq \sum_{j \in J} |(f, \xi_j)|^2 \leq B \|f\|^2
\]

Here \( A \) and \( B \) are called frame bounds. If \( A = B \), then the frame is said to be a tight frame. (Note that the family \( \{\xi_j\}_{j \in J} \) is not necessarily orthogonal, or even linearly independent.)

i) Show that if the family \( \{\xi_j\}_{j \in J} \) constitutes a tight frame, then

\[
f = A^{-1} \sum_{j \in J} (f, \xi_j) \xi_j
\]

Hint: First verify the following identity in \( H \), which will prove useful in establishing the desired result: For any \( f, g \in H \):

\[
4(f, g) = \|f + g\|^2 - \|f - g\|^2 + i\|f + ig\|^2 - i\|f - ig\|^2
\]

ii) To show that it is possible for \( \{\xi_j\}_{j \in J} \) to be a tight frame, without being orthogonal or linearly independent, consider the following (counter-)example:

\[
H = \mathbb{C}^2, \quad \xi_1 = (0, 1)^T, \quad \xi_2 = (-\frac{\sqrt{3}}{2}, -\frac{1}{2})^T, \quad \xi_3 = (\frac{\sqrt{3}}{2}, -\frac{1}{2})^T
\]

Show that the triplet \( \{\xi_1, \xi_2, \xi_3\} \) does indeed constitute a tight frame.

What is the frame bound \( A \)?
iii) Prove that, again for the general case, if \( \{\xi_j\}_{j \in J} \) is a tight frame, with frame bound \( A = 1 \), and if \( \|\xi_j\| = 1 \) \( \forall j \in J \), then the \( \xi_j \)'s constitute an orthonormal basis for \( H \).

The remaining problems in this set are all on the topic of Hilbert Spaces of Random Variables and Stochastic Processes.

34. Let \((\Omega, \mathcal{F}, \mathcal{P})\) be a probability space, and \(L_2(\Omega, \mathcal{P}; \mathbb{R}^n)\) be the Hilbert space of second-order random vectors (of dimension \( n \)) defined on \((\Omega, \mathcal{F}, \mathcal{P})\), with inner product
\[
(x, z) = E[x^T Q z]
\]
where \( Q \) is a given (fixed) positive-definite matrix of dimension \( n \times n \). Let \( \{y_0, \ldots, y_i\} \) be \( m \)-dimensional random vectors defined on \((\Omega, \mathcal{F}, \mathcal{P})\), which are uncorrelated and have zero mean. Let \( M_{nm} \) be the class of all \( n \times m \) matrices with bounded entries, and consider the following optimization problem for a given \( x \in L_2(\Omega, \mathcal{P}; \mathbb{R}^n) \):
\[
\|x - \sum_{j=0}^i \hat{K}_j y_j\| = \inf_{K_j \in M_{nm}} \|x - \sum_{j=0}^i K_j y_j\|.
\]

i) Solve for the optimal \( \hat{K}_j \), \( j = 0, \ldots, i \). Is the solution unique?

ii) Let \( \epsilon_k = \|x - \sum_{j=0}^k \hat{K}_j y_j\|^2 \), and obtain a recursive (linear first-order difference) equation for \( \epsilon_k \).

35. Let \((\Omega, \mathcal{F}, \mathcal{P})\) be a probability space, and \( x, y_1, y_2 \) three zero-mean second-order random variables defined on this space, with \( y_1 \) and \( y_2 \) uncorrelated. Let \( Z \) be the class of random variables \( z = a_1 y_1 + a_2 y_2 \), where the coefficients \( a_1 \) and \( a_2 \) are restricted to be nonnegative (that is, \( a_1 \geq 0, a_2 \geq 0 \)). We seek a best approximation to \( x \) in \( Z \) in the minimum mean square sense, that is an \( \hat{x} \in Z \) such that
\[
\inf_{z \in Z} E[(x - z)^2] = E[(x - \hat{x})^2]
\]

i) Formulate this problem as one of minimum distance to a convex set in a Hilbert space.

ii) Does there exist a unique solution? Justify your answer.

iii) Compute \( \hat{x} \) and \( E[(x - \hat{x})^2] \) when
\[
E[(y_1)^2] = E[(y_2)^2] = E[x^2] = 1, \quad E[y_1 x] = 0.2, \quad E[y_2 x] = -0.5.
\]

36. Let \((\Omega, \mathcal{F}, \mathcal{P})\) be a probability space, and \( y \) a random variable on it, with \( E[y] = 1 \) and \( E[y^2] = 2 \). We wish to find another random variable, \( x \), on the same probability space, with minimum second moment, and satisfying the constraints \( E[xy] = 2 \) and \( E[x] = -1 \).

ii) Obtain the solution if it exists.

iii) What would the solution be if the second equality constraint is replaced by the inequality constraint: $E[x] \geq -1$

37. Let $Y_1$ and $Y_2$ be uncorrelated second-order random variables defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$. Let $L_2(\Omega, \mathcal{F}; C[0,1])$ be the space of all parametrized (in $t$) random variables (equivalently, stochastic processes) $X(t;\omega)$, where for fixed $t \in [0,1]$, $X(t;\cdot)$ is a second-order random variable on $(\Omega, \mathcal{F}, \mathcal{P})$ and for fixed $\omega \in \Omega$, $X(\cdot;\omega) \in C[0,1]$. Define the inner product on $L_2(\Omega, \mathcal{F}; C[0,1])$ by

$$(X, Z) = E\left[ \int_0^1 X(t;\omega) Z(t;\omega) w(t) \, dt \right];$$

where $w(\cdot) > 0$ is in $C[0,1]$. Determine a stochastic process $\hat{X}(t;\omega) \in L_2(\Omega, \mathcal{F}; C[0,1])$ which has minimum norm and satisfies the equalities:

$$E\left[ \int_0^1 \hat{X}(t;\omega) k_i(t) Y_i(\omega) \, dt \right] = c_i, \quad i = 1, 2,$$

where $k_1, k_2$ are linearly independent elements out of $C[0,1]$, and $c_1, c_2$ are given constants.

38. Let $X$ be a second-order random variable defined on a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and $Y(t;\omega)$ be a second-order stochastic process defined on the same probability space, with $t \in [0,2]$, which is correlated with $X$, with the cross-correlation function given by $R_{XY}(t) = E[XY(t)]$. Further let $R_{YY}(t,s)$ denote the auto-correlation function of $Y$. We are interested in finding a linear least squares (l.l.s.) estimate of $X$ given the measurement process $Y(t;\omega)$ over the interval $[0,2]$, that is an estimate in the form

$$m(\omega) = \int_0^2 K(t) Y(t;\omega) \, dt$$

for some function $K(\cdot)$.

i) Show that there exists a unique such l.l.s. estimate, and obtain the equation satisfied by a corresponding optimum $K(\cdot)$ in terms of $R_{XY}$ and $R_{YY}$. Under what conditions is the optimum $K(\cdot)$ unique?

ii) Redo (i) above when $K(\cdot)$ is restricted to be a constant (independent of time).