Solution Set 5

44. Let \( S := \{ s_1 = g_1(x), s_2 = g_2(x), \ldots, s_n = g_n(x) : x \in X \} \) which is a subset of \( \mathbb{R}^n \). It is in fact a subspace, because \( \theta \in S \), and \( y, z \in S \), \( \alpha, \beta \in \mathbb{R} \Rightarrow \alpha y + \beta z \in S \). Define a linear functional, \( F \), on \( S \) by \( F(s) = f(x) \), which is clearly linear (because \( f \) and \( g_i \)'s are, and \( g_i(x) = 0 \forall i \Rightarrow f(x) = 0 \)), and is bounded (because its domain is finite dimensional). By the Hahn-Banach Theorem (extension form), \( F \) can be extended from \( S \) to \( \mathbb{R}^n \). Since the dual of \( \mathbb{R}^n \) is itself, this extension, say \( y^* \), can be uniquely identified with a vector \( \lambda = (\lambda_1, \ldots, \lambda_n)' \), that is
\[
< y, y^* > = y^*(y) = \sum_{i=1}^{n} \lambda_i y_i
\]
Also, since \( y^* \) agrees with \( F \) on \( S \),
\[
< s, y^* > = \sum_{i=1}^{n} \lambda_i s_i = \sum_{i=1}^{n} \lambda_i g_i(x) = F(s) = f(x)
\]
and this establishes the desired result.

45. The problem is:
\[
\min_{x \in X} \int_{0}^{2} |x(t)|^3 \, dt \quad \text{such that} \quad \int_{0}^{2} t^2 x(t) \, dt = 2
\]
where \( X = L_3[0,2] \equiv Z^* \), \( Z = L_{3/2}[0,2] \). Hence we have a minimum norm problem in a dual space \( (Z^*) \), subject to an equality constraint : \( \min_{<t^2,z^*>=2} \|z^*\| \). By the theory developed in class (Correspondence 11),
\[
\min_{<t^2,z^*>=2} \|z^*\| = 2 \max_{\|at^2\|_{3/2} \leq 1} a = 2 \max_{|2^{1/3}a| \leq 1} a = 2^{-\frac{2}{3}}.
\]
The minimizing solution is obtained from the condition that \( z_o^* \) is aligned with \( at^2 \):
\[
z_o^* = \alpha \text{sgn}(t) ((t^2)^{\frac{3}{2}})^{\frac{1}{3}} = \alpha t \quad \text{where} \quad \alpha \int_{0}^{2} t^3 \, dt = 2 \Rightarrow \alpha = \frac{1}{2}.
\]
Hence, \( x_o(t) = (1/2)t \). The solution is unique, because the alignment condition in \( Z^* \) is both necessary and sufficient.

46. Let \( D_T = \{ x \in X_T : \int_{0}^{T} x(t) \, dt = -3; \int_{0}^{T} tx(t) \, dt = 0 \} \), where \( T > 0 \) is fixed and \( X_T \) is a real linear normed space to be delineated later.
Let \( I = \{ T > 0 : \exists x \in D_T, |x(t)| \leq 1 \forall 0 \leq t \leq T \} \).
Then, the problem is to find the infimum of \( I \), to show that it exists, and to find the corresponding solution \( x \). I now show that this problem is equivalent to the following 2-stage minimum norm problem.
Problem A: For each fixed \( T > 0 \), let \( Z_T = L_1[0,T] \) and \( X_T = Z_T^* = L_\infty[0,T] \), and consider the two problems given below:

1. \( d_T = \min_{x \in D_T} \|x\| = \min_{z^* \in D_T} \|z^*\| = \|x^*_T\| \) (if it exists)
2. \( \inf\{T > 0 : \|x^*_T\| \leq 1\} = T_o \) (if it exists).

(a) Proposition: In Problem A, there exists a solution to both 1 and 2, and \((T_o, x_{T_o}^*)\) also constitutes a solution to the original problem.

Proof:

i) Solution to the first norm minimization problem will exist by Corollary 1 (p. 123) since \( X_T \) is dual to a normed linear space and the constraints are linear. I now claim that the distance \( d_T \) is a nonincreasing function of \( T \). To show this, assume that, to the contrary, \( \exists T_1, T_2, 0 < T_1 < T_2 \) for which \( d_{T_1} < d_{T_2} \). Let the optimal solution corresponding to \( T_1 \) be \( x_1(t), 0 \leq t \leq T_1 \).

Let \( \bar{x}_2(t) = \left( \begin{array}{c} x_1(t) \ 0 \leq t \leq T_1 \\ 0 \ T_1 < t \leq T_2 \end{array} \right) \)
which clearly belongs to \( L_\infty[0,T_2] \) whenever \( x_1 \in L_\infty[0,T_1] \). Furthermore, \( \bar{x}_2 \in D_{T_2} \), i.e., it satisfies the constraints (because \( x_1 \in D_{T_1} \)). However, the strict inequality

\[
\|\bar{x}_2\|_{[0,T_2]} = d_{T_1} < d_{T_2} = \min_{x \in D_{T_2}} \|x\|_{[0,T_2]}
\]

contradicts with the minimality of \( d_{T_2} \).

ii) Existence of an infimizing \( T \) follows because, as we will show later when we compute \( d_T \), \( d_T \) is continuous in \( T \) and the set \( \{T > 0, d_T \leq 1\} \) is closed and bounded. [We will in fact see that the infimizing \( T, T_o \), is unique]. To prove that this \( T_o \) is indeed a solution to the original problem, assume that, to the contrary, \( \exists T_o, 0 < T < T_o \), such that \( \exists x(t), t \in [0,T], x \in D_T, \|x\|_{[0,T]} \leq 1 \). But this implies that

\[
d_T \leq 1
\]

and since \( d_T \) is continuous, this contradicts with the minimality of \( T_o \).

(b) Now let us solve Problem A. By Corollary 1, p. 123 of the Text, for each \( T > 0 \), a solution exists in the dual space \( L_\infty(0,T) \), and it is characterized by

\[
d_T = \min_{x \in D_T} \|x\|_{[0,T]} = \max_{a_1, a_2} \int_0^T (-3a_1).
\]

Clearly, the solution exists, since the cost functional is continuous over a closed and bounded subset of \( \mathbb{R}^2 \). Furthermore, since the cost functional is linear, and the constraint set is convex, the solution has to be on the boundary. Let \( -a_1 = \alpha \). Then, we seek to solve

\[
\max_{\alpha, a_2} \alpha \quad \text{s.t.} \quad \int_0^T (| - \alpha + a_2 t|) dt = 1.
\]
If \((\alpha^\circ, a^2)\) denotes a solution, clearly \(\alpha^\circ > 0\). Furthermore, \(-\alpha + a_2t\) switches sign at most once (say at \(t_o\)) with the first sign being negative since \(\alpha > 0\). Hence,

\[
\int_0^T | - \alpha + a_2t | dt = \int_0^{t_o} (\alpha - a_2t) dt + \int_{t_o}^T (a_2t - \alpha) dt = 2\alpha t_0 - a_2 t_0^2 - \alpha T + \frac{a_2 T^2}{2} = 1.
\]

Furthermore, \(t_o = \alpha/a_2\), and substituting this in the above equation we obtain:

\[
f(\alpha, a_2) \coloneqq \frac{\alpha^2}{a_2} - \alpha T + \frac{a_2 T^2}{2} - 1 = 0
\]

as the constraint equation. Now, we have to maximize \(\alpha\), with \(\alpha\) and \(a_2\) satisfying \(f(\alpha, a_2) = 0\). The unique solution to this optimization problem is

\[
\alpha = \frac{1 + \sqrt{2}}{T}, \quad a_2 = \frac{2 + \sqrt{2}}{T^2}.
\]

Since \(d_T = 3\alpha = \frac{3 + 3\sqrt{2}}{T}\) (note that \(d_T\) is a nonincreasing—in fact decreasing—function of \(T\)), the condition that \(d_T \leq 1\) leads to

\[
T_o = 3(1 + \sqrt{2})
\]

as the smallest \(T\) that satisfies this constraint.

Hence, the optimal solution is (since \(t_0 = \frac{\alpha}{a_2} = \frac{1 + \sqrt{2}}{2 + \sqrt{2}}(3 + 3\sqrt{2}) = 3 + \frac{3\sqrt{2}}{2}\))

\[
x_o(t) = \begin{cases} -1, & 0 \leq t \leq 3 + \frac{3\sqrt{2}}{2} \\ 1, & 3 + \frac{3\sqrt{2}}{2} < t \leq 3 + 3\sqrt{2} \end{cases}
\]

47. The constraints are

\[
\int_0^1 (1 - t)x(t) \, dt = 1 \\
\int_0^1 (1 - t)y(t) \, dt = \frac{3}{2},
\]

and the optimization problem is:

\[
\text{minimize } \int_0^1 \sqrt{x^2(t) + y^2(t)} \, dt \text{ subject to (1) and (2).}
\]

Choose \(Z = C^1[0,1]\): space of all continuous functions taking values in \(\mathbb{R}^2\). Pick as a norm on this space:

\[
\|z\|_\infty = \max_t |z(t)|_2 = \max_t \sqrt{z_1^2(t) + z_2^2(t)}
\]

\[ \uparrow \quad \text{Euclidean norm for a vector in } \mathbb{R}^2. \]
The dual space is \((\text{see Problem 7, p. 138})\).

\[ Z^* = NBV^2[0, 1] : \text{space of all normalized functions with bounded variations, taking values in } \mathbb{R}^2. \]

Norm in this case is \(\|\nu\|_* = \int_0^1 \sqrt{(dv_1)^2 + (dv_2)^2}. \) By Corollary 1, p. 123, we have

\[
\alpha = \min_{z^* \in M^\perp} \|z^*\|_* = \max_{a_1, a_2 \in \mathbb{R}} a_1 c_1 + a_1 c_2 = a_1^2 c_1 + a_2^2 c_2
\]

where

\[ q_1 = \begin{pmatrix} 1 - t \\ 0 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 0 \\ 1 - t \end{pmatrix}; \quad c_1 = 1, c_2 = \frac{3}{2}. \]

Furthermore, \(\|a_1 q_1 + a_2 q_2\| = \max_{t} |a_1 q_1(t) + a_2 q_2(t)|_2 = \left( \sqrt{a_1^2 + a_2^2} \right) \max(1-t) = \sqrt{a_1^2 + a_2^2}. \)

Hence, the related finite-dimensional optimization problem is

\[
\text{maximize } (a_1 + \frac{3}{2} a_2) \quad \text{s.t. } a_1^2 + a_2^2 \leq 1.
\]

The solution is unique (we have a tangent line to a circle of radius 1) and is at the intersection of \(a_1 = \frac{2}{3} a_2\) and \(a_1^2 + a_2^2 = 1.\)

\[ \Rightarrow a_1^o = \frac{2}{\sqrt{13}}, \quad a_2^o = \frac{3}{\sqrt{13}} \Rightarrow d = a_1^o + a_2^o \frac{3}{2} = \frac{\sqrt{13}}{2} = d \]

The desired solution can be obtained from the alignment condition:

\[
z_0^o(z_0) = z_0^o(a_1^o q_1 + a_2^o q_2) = \int_0^1 \left\{ \frac{2}{\sqrt{13}} (1-t)dv_1^o(t) + \frac{3}{\sqrt{13}} (1-t)dv_2^o(t) \right\}
\]

\[
\Rightarrow \|a_1^o q_1 + a_2^o q_2\| \|dv^o\| \Rightarrow \nu_1^o(t) = \begin{cases} 0 & t = 0 \\ 1 & 0 < t \leq 1 \end{cases} \quad \nu_2^o(t) = \begin{cases} 0 & t = 0 \\ \frac{3}{2} & 0 < t \leq 1 \end{cases}
\]

Hence

\[
x^o(t) = \delta(t) \quad y^o(t) = \frac{3}{2} \delta(t) \quad \text{both are impulses at } t = 0.
\]

48. (i) We are given that \((x_n - x^o, x) \to 0\) (and equivalently that \((x, x_n - x^o) \to 0\)) for all \(x \in X\), and that \(\|x_n\| \to \|x^o\|\). Then,

\[
\|x_n - x^o\|^2 = \|x_n\|^2 - (x_n, x^o) - (x^o, x_n - x^o)
\]

\[
= \|x_n\|^2 - \|x^o\|^2 - (x_n - x^o, x^o) - (x^o, x_n - x^o) \to 0
\]
(ii) Take $x^o = 0$, without any loss of generality. By weak convergence, for every $y \in X$, $(x_n, y) \to 0$ as $n \to \infty$, which means that given $y \in X$, we can find $N > 0$ such that $|(x_n, y)| < \delta$ for all $n > N$. Clearly, also, given $y_1, y_2, \ldots, y_m \in X$ and $\delta > 0$ there exists $N > 0$ such that $|(x_n, y_i)| < \delta$ for all $n > N$, $i = 1, \ldots, m$. Now, given the sequence $\{x_n\}$, choose a subsequence $\{x_{n_k}\}$ as follows:

Choose $x_{n_1} = x_1$.

Choose $x_{n_2}$ such that $|(x_{n_1}, x_{n_2})| < 1$ (such an $x_{n_2}$ exists from the weak convergence property above).

Choose $x_{n_3}$ such that $|(x_{n_1}, x_{n_3})| < \frac{1}{2}$, $|(x_{n_2}, x_{n_3})| < \frac{1}{2}$ (again use the weak convergence property above, with $y_1 = x_{n_1}$, $y_2 = x_{n_2}$, and $\delta = \frac{1}{2}$).

Iteratively pick $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$, and choose $x_{n_{k+1}}$ such that

$$|(x_{n_i}, x_{n_{k+1}})| < \frac{1}{k}, \quad i = 1, 2, \ldots, k$$

Also, since $(x_n, x) \to 0$ for all $x \in X$, $(x_n, x) = \|x_n\|^2$ can be uniformly bounded, say by $M^2$. Then,

$$\|y_m\|^2 = \|\frac{1}{m} \sum_{k=1}^{m} x_{n_k}\|^2$$

$$\leq \left(\frac{1}{m}\right)^2 \left(mM^2 + 2 \sum_{i=2}^{m} \sum_{j=1}^{i-1} |(x_{n_j}, x_{n_i})|\right)$$

$$\leq \left(\frac{1}{m}\right)^2 \left(mM^2 + 2(m-1)\right) \to 0 \text{ as } m \to \infty$$

Hence, $y_m$ converges strongly to 0.

49. (i) We want to show that a linear functional $f$ on a normed space $X$ can be expressed in the form $f(x) = \langle x, x^* \rangle$, with $x^* \in X^*$, if and only if it is weakly continuous.

First let $f(x) = \langle x, x^* \rangle$, and in the definition of weak continuity given $\epsilon > 0$ choose $\delta = \varepsilon$ and $x_1^* = x^*$. Then,

$$|\langle x, x_1^* \rangle| < \delta \implies |f(x)| < \epsilon$$

and hence $f$ is weakly continuous at $x = \theta$ and thereby everywhere (since $f$ is linear). Note that weak continuity is a stronger notion of continuity than regular continuity, in the sense that weak continuity implies continuity in norm, but not vice versa.

We now prove the converse. We are given that $f$ is weakly continuous, say at $x_o = \theta$ without any loss of generality. This implies that given $\varepsilon > 0$, there exits $\delta > 0$ and a finite collection $\{x_1^*, x_2^*, \ldots, x_n^*\}$ from $X^*$ such that $|f(x)| < \epsilon$ for all $x$ satisfying $\langle x, x_i^* \rangle < \delta$, $i = 1, \ldots, n$. In particular, $|f(x)| < \epsilon$ for all $x$ such that $\langle x, x_i^* \rangle = 0$ for all $i$. We now claim that for all such $x$ it is necessary that $f(x) = 0$. Assume the contrary. Then, there exists $r > 0$ such that $f(x) = r$ for some $x$ with the property $\langle x, x_i^* \rangle = 0$ for all $i$, say $x^o$. Let $y := \frac{1+r}{r} x^o$, which clearly satisfies $\langle y, x_i^* \rangle = 0$ for all $i$. But, using the linearity property of $f$, $f(y) = 1+\epsilon > \epsilon$,
thus contradicting the initial hypothesis that $|f(x)| < \epsilon$ for all such $x$. Therefore, we have necessarily: $f(x) = 0$ for all $x$ such that $<x, x_i^*> = 0$ for all $i$.

Now, using the result of Problem 44, there exist scalars $\lambda_1, \ldots, \lambda_n$, such that

$$f = \sum_{i=1}^{n} \lambda_i x_i^* \quad \Rightarrow \quad f(x) = <x, x^*> \quad \text{with} \quad x^* := \sum_{i=1}^{n} \lambda_i x_i^*$$

(ii) Here we want to show that a linear functional $g$ on the dual space $X^*$ can be expressed in the form $g(x^*) = <x, x^*>$, with $x \in X$, if and only if it is weak$^*$ continuous. Again, it will be sufficient to consider weak$^*$ continuity only at the origin, $x^* = \theta$; that is given $\epsilon > 0$, there exist a finite collection \{x_1, x_2, \ldots, x_n\} and a $\delta > 0$, such that $g(x^*) < \epsilon$ for all $x^*$ satisfying $<x_i, x^*> < \delta$, $i = 1, \ldots, n$. This leads to, as in part (i), $|g(x^*)| < \epsilon$ for all $x^*$ such that $<x_i, x^*> = 0$ for all $i$. Picking $\delta = \epsilon$ and $x^* = x_i^*$ immediately leads to weak$^*$ continuity of $g(x^*)$.

Proof of the converse is also very similar to that in part (i) above. Similar reasoning leads to: $g(x^*) = 0$ for all $x^*$ such that $<x_i, x^*> = 0$ for all $i$. Again using the result of Problem 44, this time to linear functionals defined on $X^*$, leads to

$$g \sum_{i=1}^{n} \lambda_i x_i \quad \Rightarrow \quad g(x^*) = <x, x^*> \quad \text{with} \quad x := \sum_{i=1}^{n} \lambda_i x_i$$