Solution Set 4

33. i) Let

\[ M = \left\{ z \in L_2(\Omega, \mathcal{P}, \mathbb{R}^n) : z = \sum_{j=0}^{i} K_j y_j, \quad K_j \in \mathcal{M}_{nm} \right\}. \]

This is a closed linear subspace of \( L_2(\Omega, \mathcal{P}; \mathbb{R}^n) \) which is a Hilbert space. Hence, from the Projection Theorem, there exists a unique \( \hat{x} \in M \), expressed as

\[ \hat{x} = \sum_{j=0}^{i} \hat{K}_j y_j, \]

with

\[ \inf_{K_j \in \mathcal{M}_{nm}} \| x - \sum_{j=0}^{i} K_j y_j \| = \| x - \hat{x} \| \]

and a necessary and sufficient condition for \( \hat{x} \) to be the minimizing solution is

\[ (x - \hat{x}, z) = 0 \quad \forall z \in M \]

\[ \iff \quad E[x^T Q K_\ell y_\ell] = \sum_{j=0}^{i} E[y_j^T \hat{K}_j^T Q K_\ell y_\ell] \quad \forall K_\ell \in \mathcal{M}_{nm} \quad \ell = 0, 1, \ldots, i \]

\[ \iff \quad \text{Tr} \left\{ E[y_\ell x^T Q K_\ell] \right\} = \sum_{j=0}^{i} \text{Tr} \left\{ E[y_j y_j^T \hat{K}_j^T Q K_\ell] \right\} \quad \forall K_\ell \in \mathcal{M}_{nm}. \]

Since \( Q \) is positive definite, \( Q K_\ell \in \mathcal{M}_{nm} \) whenever \( K_\ell \in \mathcal{M}_{nm} \), and vice versa, and hence the earlier condition is

\[ \text{Tr} \left\{ \left( \Lambda_{\ell x} - \sum_{j=0}^{i} \Lambda_{\ell j} \hat{K}_j^T \right) K \right\} = 0 \quad \forall K \in \mathcal{M}_{nm} \quad \ell = 0, 1, \ldots, i \]

where

\[ \Lambda_{\ell x} \triangleq E[y_\ell x^T] ; \quad \Lambda_{\ell j} \triangleq E[y_j y_j^T]. \]

Now, two random vectors are uncorrelated if their components (considered as random variables) are uncorrelated. Furthermore, since \( E[y_\ell] = 0 \), we have

\[ \Lambda_{\ell j} = \begin{cases} \Lambda_{\ell \ell} & j = \ell \\ 0 & \text{otherwise}. \end{cases} \]

Hence, the condition now becomes

\[ \text{Tr} \left\{ \left( \Lambda_{\ell x} - \Lambda_{\ell \ell} \hat{K}_\ell^T \right) K \right\} = 0 \quad \forall K \in \mathcal{M}_{nm}, \quad \ell = 0, \ldots, i; \]
and assuming that $\Lambda_{\ell \ell}$ is invertible ($\ell = 0, \ldots, i$), we have a unique solution

$$\hat{K}_\ell = \Lambda_{\ell x}^T \Lambda_{\ell \ell}^{-1}, \quad \ell = 0, 1, \ldots, i.$$ 

**Note:** Uniqueness of the $\hat{K}_j$'s follows from the fact that $\text{Tr} [AB] = 0 \forall B \Rightarrow A$ is a matrix with only zero entries (i.e., the zero matrix).

If $\Lambda_{\ell \ell}$ is singular, then the solution is still unique in $M$ (which follows from the Projection Theorem, as stated earlier), but the corresponding $\hat{K}_j$'s may not be unique. They will then be solved from the equations:

$$\hat{K}_\ell \Lambda_{\ell \ell} = \Lambda_{\ell x}^T, \quad \ell = 0, 1, \ldots, i.$$ 

**ii)**

$$\epsilon_k = \left\| x - \sum_{j=0}^{k-1} \hat{K}_j y_j - \hat{K}_k y_k \right\|^2 = \left\| x - \sum_{j=0}^{k-1} \hat{K}_j y_j \right\|^2 + \left\| \hat{K}_k y_k \right\|^2 - 2 \left( x - \sum_{j=0}^{k-1} \hat{K}_j y_j, \hat{K}_k y_k \right)$$

$$= \epsilon_{k-1} + \text{Tr} \left[ \Lambda_{kk} \hat{K}_k^T Q \hat{K}_k \right] - 2 \text{Tr} \left\{ E \left[ y_k x^T Q \hat{K}_k \right] \right\}$$

$$= \epsilon_{k-1} + \text{Tr} \left[ \Lambda_{kk} Q \Lambda_{kk}^T \Lambda_{kk}^{-1} \right] - 2 \text{Tr} \left\{ \Lambda_{kk} Q \Lambda_{kk}^T \Lambda_{kk}^{-1} \right\}$$

$$\Leftrightarrow \epsilon_k = \epsilon_{k-1} - \text{Tr} \left[ \Lambda_{kk} Q \Lambda_{kk}^T \Lambda_{kk}^{-1} \right].$$

**34. i)** Note that $Z$ is a closed, convex subset of the Hilbert space $L_2(\Omega, P; \mathbb{R})$ (closed, because $a_1 \geq 0, a_2 \geq 0$, and convex, because the set of $(a_1, a_2) \in \mathbb{R}^2$, $a_1 \geq 0, a_2 \geq 0$, is convex). Then the problem is one of minimizing $\|x - z\|$ over $z \in Z$, or (equivalently) one of finding the minimum distance from $x \in L_2(\Omega, P; \mathbb{R})$ to $Z$.

**ii)** It follows from Theorem 1, p. 69 of Luenberger that there exists a unique $\hat{x} \in Z$ that minimizes $\|x - z\|$ on $Z$. Furthermore,

$$(x - \hat{x}, z - \hat{x}) \leq 0 \quad \forall z \in Z$$

**iii)** Writing $\hat{x}$ and $z \in Z$ as $\hat{x} = \hat{a}_1 y_1 + \hat{a}_2 y_2$ and $z = a_1 y_1 + a_2 y_2$, the condition in part (ii) above becomes

$$(x - \hat{a}_1 y_1 - \hat{a}_2 y_2, (a_1 - \hat{a}_1)y_1 + (a_2 - \hat{a}_2)y_2) = 2(0.3 - \hat{a}_1)(a_1 - \hat{a}_1) - (20.4 + \hat{a}_2)(a_2 - \hat{a}_2) \leq 0 \quad \forall a_1 \geq 0, a_2 \geq 0$$

Pick $a_1 = \hat{a}_1 \Rightarrow 2(0.4 + \hat{a}_2)(a_2 - \hat{a}_2) \geq 0 \forall a_2 \geq 0 \Leftrightarrow \hat{a}_2 = 0$

Now pick $a_2 = \hat{a}_2 = 0 \Rightarrow 2(0.3 - \hat{a}_1)(a_1 - \hat{a}_1) \leq 0 \forall a_1 \geq 0 \Leftrightarrow \hat{a}_1 = 0.3$. Hence,

$$\hat{x} = 0.3 y_1, \quad E[(x - \hat{x})^2] = \|x - \hat{x}\|^2 = \|x - 0.3 y_1\|^2 = 1.82$$

2
35. Let $H$ be the Hilbert space of second-order random variables defined on $(\Omega, \mathcal{F}, \mathcal{P})$. Then, this problem can be posed as a minimum norm problem on $H$, subject to equality constraints. That is, letting $(\cdot, \cdot)$ denote the norm on $H$,
\[
\min_{x \in H} \|x\| \quad \text{such that} \quad (x, 1) = 2 \quad \text{and} \quad (x, y) = -6
\]

\text{i) Note that $y$ and 1 are linearly independent on $H$, because if they were linearly dependent, then we would have from $E[y] = 2$, $y = 2$ with probability 1, which however contradicts with $E[y^2] = 8$. Then, from the Projection Theorem the problem admits a unique solution in $M = [1, y]$.}

\text{ii) Since the solution is in $M$, it is in the form $x = 1 + 2y$, for two real numbers $\alpha_1$ and $\alpha_2$. These can be obtained from the given two constraints:}
\[
(\alpha_1 + \alpha_2 y, 1) = 2, (\alpha_1 + \alpha_2 y, y) = -6 \quad \Rightarrow \quad \alpha_1 = 7, \alpha_2 = -5/2
\]
Note that the minimum value is $\|x^o\|^2 = \|7 - 5y\|^2 = 29$.

\text{iii) Replace the constraint $(x, 1) = 2$ above with $(x, 1) = c$ where $c \leq 2$. For each fixed $c$, the problem again admits a unique solution (by the same reasoning as above), and is given by}
\[
x^o = \alpha_1 + \alpha_2 y, \quad \alpha_1 + 2\alpha_2 c, 2\alpha_1 + 8\alpha_2 = -6 \quad \Rightarrow \quad \alpha_1 = 3 + 2c, \alpha_2 = -(3 + c)/2
\]
Then, $\|x^o\|^2 = (3 + 2c)^2 - 2c(3 + c) = 9 + 6c + 2c^2$, which is minimized uniquely by choosing $c = -3/2$ (simply find the stationary point of the strictly convex function $\|x^o\|^2$), which also satisfies the constraint $c \leq 2$. This then leads to (as the unique solution):
\[
\alpha_1 = \frac{3}{2}, \alpha_2 = -\frac{3}{4} \quad \Rightarrow \quad x^o = \frac{3}{2} - \frac{3}{4} y \quad \Rightarrow \quad \|x^o\|^2 = 9
\]
Note that the value of $\|x^o\|^2$ here is smaller than the one (29) obtained in part (ii) above; this is to be expected because under the inequality constraint there is more freedom in the choice of $x^o$.

36. Let $E[Y_i] \triangleq \bar{y}_i$, $E[Y_i^2] = r_i$, and take $w$ to be a positive function in $C[0, 2]$. Let $\tilde{k}_i(t) = k_i(t)/w(t)$. $k_1$ and $k_2$ linearly independent in $C[0, 2] \Rightarrow \tilde{k}_1$ and $\tilde{k}_2$ are linearly independent in $L_2[0, 2]$. Then, the optimization problem is:
\[
\min \|X\|; \quad X \in L_2(\Omega, \mathcal{P}; C[0, 2])
\]
such that
\[
(X, \tilde{k}_1 Y_1) = c_1; \quad (X, \tilde{k}_2 Y_2) = c_2.
\]
Let us first consider the case where $X$ is allowed to be in $L_2(\Omega, \mathcal{P}; L_2[0, 2])$. Since this is a Hilbert space, the conditions of Theorem 2, p. 65 of the text, are satisfied, implying that the solution to the second problem, say $\hat{x}$, is unique and is given by
\[
\hat{x}(t, w) = \beta_1 \tilde{k}_1(t) Y_1(\omega) + \beta_2 \tilde{k}_2(t) Y_2(\omega)
\]
where
\[
\begin{pmatrix}
\beta_1 \\
\beta_2
\end{pmatrix}
= \begin{pmatrix}
\|\bar{k}_1 Y_1\|^2 & (\bar{k}_1 Y_1, \bar{k}_2 Y_2) \\
(\bar{k}_2 Y_2, \bar{k}_1 Y_1) & \|\bar{k}_2 Y_2\|^2
\end{pmatrix}^{-1}
\begin{pmatrix}
c_1 \\
c_2
\end{pmatrix}.
\]

The matrix here is invertible, because \(\bar{k}_1\) and \(\bar{k}_2\) are linearly independent (WHY?). Since \(Y_1\) and \(Y_2\) are uncorrelated
\[
(\bar{k}_1 Y_1, \bar{k}_2 Y_2) = (\bar{k}_2 Y_2, \bar{k}_1 Y_1) = E[\int_0^2 \frac{k_1(t)}{w(t)} k_2(t) dt Y_1 Y_2]
\]
\[
= K_{12 \bar{y}_1 \bar{y}_2}, \text{ where } K_{12} = \int_0^2 \frac{k_1(t)k_2(t)}{w(t)} dt.
\]
Furthermore,
\[
\|\bar{k}_i Y_i\|^2 = r_i \int_0^2 \left[ \frac{k_i^2(t)}{w(t)} \right] dt \triangleq r_i K_i
\]
\[
\begin{pmatrix}
r_1 K_1 & K_{12 \bar{y}_1 \bar{y}_2} \\
K_{12 \bar{y}_1 \bar{y}_2} & r_2 K_2
\end{pmatrix}
\]
\[
\begin{pmatrix}
1 \\
1
\end{pmatrix}
= \frac{1}{r_1 r_2 K_1 K_2 - (K_{12 \bar{y}_1 \bar{y}_2})^2}
\begin{pmatrix}
r_2 K_1 & -K_{12 \bar{y}_1 \bar{y}_2} \\
-K_{12 \bar{y}_1 \bar{y}_2} & r_1 K_1
\end{pmatrix}
\]
\[
\therefore
\begin{align*}
\hat{\beta}_1 &= (c_1 r_2 K_2 - c_2 K_{12 \bar{y}_1 \bar{y}_2})/[r_1 r_2 K_1 K_2 - (K_{12 \bar{y}_1 \bar{y}_2})^2] \\
\hat{\beta}_2 &= (c_2 r_1 K_1 - c_1 K_{12 \bar{y}_1 \bar{y}_2})/[r_1 r_2 K_1 K_2 - (K_{12 \bar{y}_1 \bar{y}_2})^2] \\
\hat{x}(t, w) &= \hat{\beta}_1 \bar{k}_1(t) Y_1(\omega) + \hat{\beta}_2 \bar{k}_2(t) Y_2(\omega)
\end{align*}
\]

Note that \(\hat{x}\) also belongs to \(L_2(\Omega, \mathcal{P}; C[0,2])\), because \(\bar{k}_1\) and \(\bar{k}_2\) are continuous. Hence, \(\hat{x}\) is the unique vector solving the original problem.

37. i) First note that \(m\) as defined is a second-order random variable if \(K(\cdot)\) is square-integrable, because by the Cauchy-Schwarz inequality,
\[
E[m^2] \leq \int_0^1 |K(t)|^2 dt \int_0^1 E[Y^2(t)] dt
\]
and the second product term is finite. Now, the set of all \(m\)'s corresponding to such \(K\)'s is a closed subspace \(M\) of \(L_2(\Omega, \mathcal{P}; \mathbb{R})\), and hence by the Projection Theorem, there is a unique projection of \(X\) onto \(M\), and by orthogonality we have the equation
\[
R_{XY}(t) = \int_0^1 \hat{K}(s) R_{YY}(s, t) ds
\]
which has a solution \(\hat{K}\), and in terms of this,
\[
\hat{X} = \int_0^1 \hat{K}(t) Y(t) dt
\]
is the unique element of $M$. And $K$ is unique if $Y_Y(s, t)$ is positive definite.

**ii)** Now $M$ is the one-dimensional subspace generated by the random variable

$$Z(\omega) := \int_0^1 Y(t; \omega) dt.$$ 

Note that this random variable has second moment (or equivalently norm squared)

$$\sigma_Z^2 := \int_0^1 \int_0^1 R_{YY}(s, t) ds dt,$$

and

$$E[XZ] = \int_0^1 R_{XY}(t) dt =: \sigma_{XZ}.$$ 

Now we have projection of one random variable ($X$) into the subspace generated by another ($Z$), and the result follows from standard theory discussed in class:

$$\hat{X} = (\sigma_{XZ}/\sigma_Z^2)Z$$

where we have assumed without any loss of generality that $\sigma_Z^2$ is nonzero, because otherwise $R_{YY}(s, t)$ would be identically equal to zero. Clearly the solution is unique.

38. First note that by changing the orders of integration, $f$ can be re-written as

$$f(x) = \int_0^\infty ds e^{-4s} x(s) \int_s^\infty e^{4(s-t)} K(t, s) dt$$

Clearly, $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$, for all $x, y \in H$, $\alpha, \beta \in \mathbb{R}$, and hence $f$ is linear. Furthermore, $f(x) = (x, y)$, where (by inspection from the preceding expression for $f$):

$$y(s) = \int_s^\infty e^{4(s-t)} K(t, s) dt$$

and this is the unique such $y$. Let us now bound the norm of $y$:

$$\|y\|^2 = \int_0^\infty e^{-4s} |y(s)|^2 ds = \int_0^\infty e^{4s} \left( \int_s^\infty e^{-4t} K(t, s) dt \right)^2 ds$$

$$\leq \int_0^\infty e^{4s} ds \left[ \int_s^\infty e^{-4t} dt \right] \left[ \int_s^\infty |K(t, s)|^2 dt \right] = \frac{1}{8} \int_0^\infty e^{-4s} ds \int_s^\infty |K(t, s)|^2 dt$$

$$\leq \frac{1}{8} \int_0^\infty e^{-4s} ds \int_s^\infty |K(t, s)|^2 dt \leq \frac{1}{8} \int_0^\infty ds \int_0^\infty |K(t, s)|^2 dt < \infty,$$

where the inequality in the second line follows from Cauchy-Schwarz, the next equality follows from the integration of the first product term, and the first inequality in the third line follows from extending the interval of integration from $[s, \infty)$ to $[0, \infty)$, and the second bound on that line follows from the fact that $e^{-4s} \leq 1$. Hence, under the assumption that $K$ is square-integrable, $y$ is bounded in the norm of $H$, and hence $y \in H$. Thus we have

$$|f(x)| \leq M \|x\| \quad \text{where} \quad M \leq \|y\|$$

which shows that $f$ is bounded, and hence continuous. This completes both parts (i) and (ii).
39. i) $f(x) = \int_0^t dt \int_0^s K(t, s)x(s) \, ds$

\[ f(\alpha x + \beta y) = \int_0^1 dt \int_0^t K(t, s)[\alpha x(s) + \beta y(s)] \, ds = \alpha f(x) + \beta f(y) \quad \cdots \text{linear} \]

\[ |f(x)| = \left| \int_0^3 ds \int_0^t K(t, s)x(s) \, ds \right| = \left| \int_0^3 ds \int_s^1 K(t, s)dt \right| \]

\[ \leq \int_0^3 ds |x(s)| \cdot \left| \int_0^3 K(t, s) dt \right| \]

\[ \leq \left\{ \int_0^3 ds |x(s)|^4 \right\}^{\frac{1}{4}} \left\{ \int_0^3 ds \left| \int_s^3 K(t, s) dt \right|^{\frac{3}{4}} \right\}^{\frac{4}{3}} =: M \|x\|_4 \]

↑ Hölder’s inequality

Hence the condition on $K$ is that $M$ should be finite. Under this condition, $f$ is bounded, and hence continuous.

ii) It follows from the above, given also that Hölder’s inequality is tight,

\[ \|f\| = \left\{ \int_0^3 ds \left| \int_s^3 K(t, s) dt \right|^{\frac{4}{3}} \right\}^{\frac{3}{4}} \]

iii) Evaluating the integrals in (ii) above for $K(t, s) = t - s$, we obtain

\[ \|f\| = (1/2)(3/11)^{3/4}(3/11)^{1/4} = 3.8712. \]

40.

i) $f(\alpha y_1 + \beta y_2) = \int_\Omega (\alpha y_1(\omega) + \beta y_2(\omega)) \, d\mathcal{P}(\omega) = \alpha f(y_1) + \beta f(y_2)$

→ linear

$h(\alpha x_1 + \beta x_2) = \int_\Omega d\mathcal{P}(\omega) \int_0^2 b(t)[\alpha x_1(t; \omega) + \beta x_2(t; \omega)] \, dt = \alpha h(x_1) + \beta h(x_2)$

→ linear

For boundedness,

\[ |f(y)| = \left| \int_\Omega y(\omega) \, d\mathcal{P}(\omega) \right| \leq \left[ \int_\Omega |y(\omega)|^2 \, d\mathcal{P}(\omega) \right]^{\frac{1}{2}} \left[ \int_\Omega 1 \, d\mathcal{P}(\omega) \right]^{\frac{1}{2}} \]

\[ = \|y\| \]

→ bounded with $M = 1.$
\[ |h(x)| = \left| \int_{\Omega} dP(\omega) \int_{0}^{2} b(t)x(t;\omega)dt \right| = \left| \int_{0}^{2} b(t) \int_{\Omega} X(t;\omega)dP(\omega)dt \right| \]
\[ \leq \left\{ \int_{0}^{2} |b(t)|^2 dt \right\}^{\frac{1}{2}} \left\{ \int_{0}^{2} \int_{\Omega} |x(t;\omega)|^2 dP(\omega) \right\}^{\frac{1}{2}} \]
\[ \leq \|b\|_2 \|x\|_{L^2(\Omega;P;L^2[0,2])} \rightarrow \text{bounded with } M \leq \|b\|. \]

where the first two inequalities are Cauchy-Schwartz.

ii) \[ f(y) = \int_{\Omega} y(\omega) dP(\omega) = (y,1) \Rightarrow \int_{\Omega} g(\omega) dP(\omega) = 1 \text{ a.e. } P. \]
\[ h(x) = \int_{0}^{2} \int_{\Omega} dP(\omega) x(t;\omega)b(t) dt = <x,\bar{x}> \Rightarrow \int_{\Omega} \bar{x}(t;\omega) = b(t) \text{ a.e. } P. \]

where “a.e. \( P \)” stands for almost everywhere under measure \( P \).

41. i) \[ \|f\| = \sup_{\|x\| \leq 1} |f(x)| = \sup_{|x(t)| \leq 1} |2x(1/3) + 3x(1/2) + 4 \int_{0}^{1} tx(t)dt| \]
\[ = 2 + 3 + 4 \int_{0}^{1} tdt = 7 \]
which is attained by choosing \( x(t) \equiv 1 \).

ii) Let \( v = v_1 + v_2 + v_3 \), where
\[ 2x(1/3) = \int_{0}^{1} x(t)dv_1(t) \]
\[ 3x(1/2) = \int_{0}^{1} x(t)dv_2(t) \]
\[ 4 \int_{0}^{1} tx(t)dt = \int_{0}^{1} x(t)dv_3(t). \]

Clearly,
\[ v_1(t) = \begin{cases} 0 & 0 \leq t < 1/3 \\ 2 & 1/3 \leq t \leq 1 \end{cases} \]
\[ v_2(t) = \begin{cases} 0 & 0 \leq t < 1/2 \\ 3 & 1/2 \leq t \leq 1 \end{cases} \]
\[ \frac{dv_3(t)}{dt} = 4t, \quad v_3(0) = 0 \Rightarrow v_3(t) = 2t^2. \]
Hence,

\[
v(t) = \begin{cases} 
2t^2, & 0 \leq t < 1/3 \\
2 + 2t^2, & 1/3 \leq t < 1/2 \\
5 + 2t^2, & 1/2 \leq t < 1 
\end{cases}
\]

Since \( v \), given above, is a monotonically nondecreasing function, its total variation is \( v(1) - v(0) = 7 \), which is clearly equal to \( \|f\| \) in part (i).

42. i) First note that \( \|x\| = \max\{1, 1, \frac{2}{3}, \frac{1}{2}, \cdots\} = 1 \). Furthermore, since \( c^*_0 = \ell_1 \), \( f \) sought is in \( \ell_1 \). It should be aligned with \( x \), as the given constraint can be written as \( f(x) = \|f\| \|x\| \).

A solution exists by Corollary 2 to the Hahn-Banach theorem, and by the alignment condition \( f = \{a_i\}_{i=1}^{\infty} \) will have nonzero elements in the sequence only for \( i = 1 \) and 2. In view of this, we rewrite the constraint as:

\[
\|f\| = \sum_{i=1}^{\infty} |a_i| = |a_1| + |a_2| = 1 \quad (\ast)
\]

Also, \( f(x) = -a_1 + a_2 = \|x\| = 1 \imp a_2 - a_1 = 1 \quad (\ast\ast) \)

Then, from (\ast) and (\ast\ast) the complete solution follows, which is clearly nonunique:

\( f = \{a_i\}_{i=1}^{\infty}; \quad a_2 = 1 + a_1, \quad a_1 \leq 0, \quad a_2 \geq 0, \quad a_i = 0, \quad i \geq 3 \).

ii) Again note that \( c^*_0 = \ell_1 \). By inspection, a solution is:

\[ y^* = \{b_i\}_{i=1}^{\infty}, \quad b_1 = b_2 = \frac{1}{2}, \quad b_j = 0, \quad j \geq 3. \]

Clearly, \( <x, y^*> = 0 \) and \( \|y^*\| = 1 \).

43. We want to find two functions \( v \) and \( w \), of bounded variation and unit norm, such that

\[
x^*(x) = \int_0^3 x(t) \, dv(t) = \max_t |x(t)| = 2, \quad y^*(x) = \int_0^3 x(t) \, dw(t) = 0
\]

The solution is nonunique in each case. For the first one, any \( v \) that has positive jump discontinuities at points where \( x(t) = 2 \), and a possible negative jump discontinuity at \( t = 3/2 \), with the algebraic sum of these discontinuities equal to 1, is a solution. One of the simplest such functions is the one that is zero for \( 0 \leq t < 1/4 \), and 1 for \( 1/4 \leq t \leq 3 \). In the second case, a possible candidate is the piecewise-constant function \( w \), with \( w(0) = 0 \), and which has a single jump of 1 at the point \( t = 1 \) (or \( t = 2 \)); the solution does not have to be piecewise constant, it could even be differentiable.