Notes On

HAHN-BANACH THEOREM

Extension Form

I start with a definition – that of a sublinear functional.

**Definition.** Let $X$ be a real vector space (not necessarily normed). A map $p : X \to \mathbb{R}$ is called a sublinear functional if it satisfies the following two properties:

$$
\begin{align*}
p(x + y) &\leq p(x) + p(y) \quad \forall x, y \in X \quad \text{(subadditive)} \\
p(\alpha x) &= \alpha p(x) \quad \forall \alpha \geq 0, \forall x \in X \quad \text{(positive homogeneity)}
\end{align*}
$$

Note that “norm” is a sublinear functional.

The next result is a useful property of sublinear functionals.

**Lemma.** Let $M \subset X$ be a subspace, and $f$ a linear functional on $M$, such that for some sublinear functional $p$,

$$
 f(x) \leq p(x) \quad \forall x \in M.
$$

Let $x_0$ be a fixed element of $X$. Then, for any real number $c$, the following are equivalent:

$$
 f(x) + \lambda c \leq p(x + \lambda x_0) \quad \forall x \in M, \lambda \in \mathbb{R} \quad (1)
$$

$$
 -p(-x - x_0) - f(x) \leq c \leq p(x + x_0) - f(x) \quad \forall x \in M \quad (2)
$$

Furthermore, there is a real number $c$ satisfying (2) and hence (1).

**Proof:** To show that (1) $\Rightarrow$ (2), first set $\lambda = 1$, and then set $\lambda = -1$ and replace $x$ by $-x$; then (2) is immediate. To show the converse, first take $\lambda > 0$, and replace $x$ by $x/\lambda$ on the RHS inequality, to obtain (1). Now take $\lambda < 0$, and replace $x$ by $x/\lambda$ on the LHS inequality, to arrive again at (1). For $\lambda = 0$, (1) is always satisfied, from the definitions of $f$ and $p$. To produce the desired $c$, let $x, y$ be arbitrary elements out of $M$. Then,

$$
 f(x) - f(y) = f(x - y) \leq p(x - y)
$$

$$
 \leq p(x + x_0) + p(-y - x_0) \quad \text{by subadditivity}
$$

$$
 \Rightarrow -p(-y - x_0) - f(y) \leq p(x + x_0) - f(x) \quad \forall x, y \in M
$$

$$
 \Rightarrow \sup_{y \in M} (-p(-y - x_0) - f(y)) \leq \inf_{x \in M} \{p(x + x_0) - f(x)\}
$$

Any $c$ between the “sup” and “inf” values will do it. \hfill \diamond

Now we have the main theorem.

**Theorem (Hahn-Banach).** Given a real linear vector space $X$ and a subspace $M$, let $p$ be a sublinear functional on $X$, and $f$ a linear functional on $M$ such that $f(x) \leq p(x)$ \quad $\forall x \in M$. Then, there is a linear functional $F$ on $X$ such that $F = f$ on $M$ and $F \leq p$ on all of $X$.

**Proof.** Let $x_0 \not\in M, x_0 \in X$, and consider the subspace $M_0$ consisting of all elements $x + \lambda x_0, x \in M, \lambda \in \mathbb{R}$. We may extend $f$ to a linear functional on $M_0$ by defining

$$
 f_0(x + \lambda x_0) = f(x) + \lambda c
$$
where $c$ is any real number. Now choose $c$ such that it satisfies condition (2) in the previous Lemma. Then,

$$f_0(x + \lambda x_0) \leq p(x + \lambda x_0), \quad \iff f_0 \leq p \quad \text{on } M_0$$

Hence, we have seen how to extend $f$ from $M$ to $M_0$, which is of dimension one higher than that of $M$ (assuming that $M$ is finite-dimensional).

1. If $X$ is normed and separable, then we can find a countable set of vectors $\{x_0, x_1, \ldots\}$, all linearly independent and not in $M$, so that every element in $X$ can be approximated to any degree by linear combinations of vectors out of $M$ and $\{x_0, x_1, \ldots\}$. Let $M_i = M_{i-1} + \lambda x_i$, which is a subspace. If $y \in M_i$, we can write it as $y = x + \lambda x_i$, where $x \in M_{i-1}$ and $\lambda \in \mathbb{R}$. Define

$$f_i(x + \lambda x_i) = f_{i-1}(x) + \lambda c_i$$

where $c_i$ is as in the Lemma with $f$ replaced by $f_{i-1}$. This then shows how we can extend $f$ recursively to a countable dense subset of $X$. Call this extension $g$, which is naturally linear. By construction, $g \leq p \quad \forall x \in S$, where $S$ is a dense subset of $X$. Now, to extend $g$ to $X$, let $x \in X$ be given and $\{s_n\}$ be a sequence out of $S$ converging to $x$. Then, the limit $\lim_{n \to \infty} g(s_n)$ is well defined; call this $F(x)$. This defines $F$ pointwise. $F$ is obviously linear. Furthermore, since $g(s_n) \leq p(s_n) \quad \forall s_n \in S$, we have

$$F(x) \leq p(x) \quad \forall x \in X.$$  

2. If $X$ is not a separable normed space, we cannot find an ordered set of vectors to form a basis for a dense subset. Then, the proof will have to be modified, where the strict ordering is replaced by partial ordering, and Zorn’s Lemma is used. Zorn’s Lemma says that: “If $P$ is a partially ordered set in which every chain has an upper bound, then $P$ possesses a maximal element.” This is equivalent to the Axiom of Choice, which says the following: “Given a nonempty class of disjoint nonempty sets, a set can be formed which contains precisely one element taken from each set in the given class.” To use this in our proof, let $C$ be the collection of all pairs $(h, H)$ where $h$ is an extension of $f$ to the subspace $H \supseteq M$, and $h \leq p$ on $H$. Partially order $C$ by $(h_1, H_1) \leq (h_2, H_2)$ iff $H_1 \subseteq H_2$ and $h_1 = h_2$ on $H_1$. Then, every chain in $C$ has an upper bound (consider the union of all subspaces in the chain). This implies by Zorn’s Lemma that $C$ has a maximal element. Call this $(F, F)$. If $F \neq X$, then this means that we can extend $F$ to a larger subspace – but this contradicts its maximality. Hence $F$ is defined on $X$.  

The two corollaries below follow from the Hahn-Banach theorem. I provide a proof for only the first one.

**Corollary 1.** Let $f$ be a continuous linear functional defined on a subspace $M$ of $X$, a normed linear space. Then, there exists $F \in X^*$ such that $\|f\|_M = \|F\|_X$ and $F$ is an extension of $f$.

**Proof.** Take $p(x) = \|f\|_M \|x\|_X$ in the Hahn-Banach Theorem, which is clearly a sublinear functional. Now note that

$$F(x) \leq p(x) = \|f\|_M \|x\|_X$$

and

$$-F(x) = F(-x) \leq \|f\|_M \|x\|_X \quad \forall x \in X,$$

which leads to

$$|F(x)| \leq \|f\|_M \|x\|_X \Rightarrow \sup_{x \in X, \|x\| \leq 1} |F(x)| \leq \|f\|_M.$$

But since $\sup_{x \in X, \|x\| \leq 1} |F(x)| \geq \sup_{x \in M, \|x\| \leq 1} |F(x)| = \|f\|_M$, we actually have equality above. Since $F$ is bounded, it is also continuous.

**Corollary 2.** Let $x_0 \in X$, a normed linear space. Then, there exists a nonzero bounded linear functional $F$ on $X$ such that

$$F(x_0) = \|F\| \|x_0\|.$$  

The converse of this last result does not generally hold; that is, given $F \in X^*$, we may not be able to find $x \in X$ such that $F(x) = \|F\| \|x\|$. See, for example, Example 1 on page 113 of the text by Luenberger. We will see (in class) that for some (but not all) Banach spaces this converse indeed holds.

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