ASSIGNMENT 5

Reading Assignment:  Text: Chapters 5 & 6
Advance Reading:  Text: Chapter 7.
Recommended Reading:  Curtain & Pritchard: pp. 65-67, and Chapter 12;

Problems (to be handed in):  Due Date:  Wednesday, November 9.

44. Let $g_1, g_2, \ldots, g_n$ be linearly independent linear functionals on a vector space $X$. Let $f$ be another linear functional on $X$ such that for every $x \in X$ satisfying $g_i(x) = 0$, $i = 1, 2, \ldots, n$, we have $f(x) = 0$. Show that there exist constants $\lambda_1, \lambda_2, \ldots, \lambda_n$ such that

$$f = \sum_{i=1}^{n} \lambda_i g_i.$$ 

Hint: Use the Hahn-Banach Theorem (extension form; Correspondence 9).

45. Find a continuous function $x_o$ that minimizes the functional

$$\int_{0}^{2} |x(t)|^3 \, dt$$

subject to the constraint

$$\int_{0}^{2} t^2 x(t) \, dt = 2.$$ 

Identify the space(s) in which you work, and also indicate whether the solution you obtained is unique or not.

46. We wish to find a function $x$, satisfying the pointwise bound $|x(t)| \leq 1$ (that is, $\|x\|_\infty \leq 1$), and integral constraints

$$\int_{0}^{T} x(t) \, dt = -3 \quad \text{and} \quad \int_{0}^{T} t \, x(t) \, dt = 0$$

for the smallest possible $T > 0$.
(Hence, you have to find the optimum value of $T$, say $T_o$, and a corresponding $x(t)$ on $[0, T_o]$.)

Hint: Show that the problem can be reformulated as one of first minimizing $\|x\|_\infty$ subject to the given integral constraints and with $T$ fixed (say this solution is $x_T$), and then finding the
smallest value of $T$ such that $\|x_T\|_\infty \leq 1$. Material in Section 5.9 of the Text could be useful here.

47. **Find** two functions $x$ and $y$ that minimize the functional

$$\int_0^1 \left[ x^2(t) + y^2(t) \right]^{1/2} \, dt$$

while satisfying the equality constraints

$$\int_0^1 (1 - t) x(t) \, dt = 1 \quad \text{and} \quad \int_0^1 (1 - t) y(t) \, dt = \frac{3}{2}$$

**Note:** Part of the problem/question is to identify the space where the solution should be sought. See also Problem 7, page 138 of the Text.

48. Let $X$ be a Hilbert space, and $\{x_n\}$ be a sequence in $X$, converging weakly to $x^0 \in X$.

i) **Show** that the convergence is also in the strong sense (that is in norm) if further the sequence of real numbers $\{\|x_n\|\}$ converges to the norm of $x^0$, $\|x^0\|$.

ii) Again for the original problem, **show** that one can find a subsequence $\{x_{n_k}\}$ such that the sequence of arithmetic means

$$y_m = \frac{1}{m} \sum_{k=1}^m x_{n_k}, \quad m = 1, 2, \ldots$$

converges strongly to $x^0$ (that is, $\|y_m - x^0\| \to 0$).

49. Let $X$ be a real normed linear space, and $X^*$ be its dual.

i) **Show** that a linear functional $f$ on $X$ is weakly continuous if and only if it is of the form $f(x) = \langle x, x^* \rangle$, for some $x^* \in X^*$.

ii) **Show** that a linear functional $g$ on $X^*$ is weak* continuous if and only if it is of the form $g(x^*) = \langle x, x^* \rangle$, for some $x \in X$.

**Hint:** Use the result of Problem 44.