ASSIGNMENT 2

Reading Assignment: Text: Chp 10 (pp. 272-277); Correspondence # 3.

Recommended Reading: Curtain & Pritchard: Chapters 1 (pp. 20-22), 4 (pp. 55-64); Balakrishnan: Chapter 1, and Chapter 2 (pp. 54-57). Liusternik & Sobolev: Chapter 1 (pp. 26-44).

Advance Reading: Text: Chapter 3

Problems (to be handed in): Due Date: Wednesday, September 21
To be handed in to my secretary next door, by 4 pm.

13. Consider the space $\ell'_p$ consisting of all ordered numbers $(\xi_1, \xi_2, \ldots, \xi_{k_1})$ where $k_1$ is a natural number and $\xi_i$’s are arbitrary real numbers. If

$$x := (\xi_1, \xi_2, \ldots, \xi_{k_1}) \quad y := (\nu_1, \nu_2, \ldots, \nu_{k_2}), \quad k_2 \geq k_1,$$

we introduce a metric by

$$\rho(x, y) = \left( \sum_{i=1}^{k_1} |\xi_i - \nu_i|^p + \sum_{j=k_1+1}^{k_2} |\nu_j|^p \right)^{1/p}, \quad 1 \leq p < \infty.$$

Show that $\ell'_p$ is not a complete metric space.

Hint: You have to show that there exists a Cauchy sequence in $\ell'_p$, whose limit is not in $\ell'_p$, consider, for instance, the sequence:

$$x_1 = \{1\}, \quad x_2 = \{1, \frac{1}{2}\}, \quad x_3 = \{1, \frac{1}{2}, \frac{1}{2^2}\}, \ldots, \quad x_n = \{1, \frac{1}{2}, \ldots, \frac{1}{2^{n-1}}\}, \ldots$$

14. Let $f : C[0, 2] \to \mathbb{R}$ be a functional defined by

$$f(x) = \max_{0 \leq t \leq 2} x(t)$$

(a) Show that $f$ is continuous.

(b) Is $f$ uniformly continuous?

Note: A transformation $T : X \to Y$, where both $X$ and $Y$ are normed linear spaces, is uniformly continuous, if in the $(\epsilon, \delta)$ definition of continuity, $\delta$ depends on only $\epsilon$, and not on $x \in X$. 

15. This problem is similar to the preceding one with a different interval and a different function \( f \). Let \( f : C[0, 4] \to \mathbb{R} \) be a functional defined by

\[
f(x) = \max_{0 \leq t \leq 4} (x(t))^3
\]

(a) Show that \( f \) is continuous.
(b) Is \( f \) uniformly continuous?

16. As discussed in class, the Banach space \( L_p[a, b] \) is separable, for \( 1 \leq p < \infty \) and for any finite interval \([a, b]\) (see also Example 4, on page 43 of the text). This result does not hold, however, if the interval is infinite, that is \((\infty, \infty)\), and the norm adopted is

\[
\|x\| = \left( \lim_{T \to \infty} \frac{1}{T} \int_{-T}^{T} |x(t)|^p \, dt \right)^{\frac{1}{p}}
\]

We denote the space in this case by \( L_p(\infty, \infty) \).

Construct a counterexample to show that \( L_2(\infty, \infty) \) is not separable.

**Hint:** Try the continuum of elements \( \{\sin \alpha t, \ \alpha \in (-\infty, \infty)\} \) as a candidate to generate a counterexample.

The next five problems are on fixed points (topic of Correspondence # 3).

17. Consider the fixed-point (FP) equation

\[
x(t) = \frac{1}{2} t^3 + \alpha \sin \pi x(t)
\]

defined over the interval \( t \in [-2, 2] \), with \( x \in C[-2, 2] \), and \( \alpha \) a positive constant (a parameter). For what values of \( \alpha \) does there exist a **unique** continuous function \( x(\cdot) \) on \([-2, 2]\) which solves the FP equation. Show (prove) that for these values of \( \alpha \) a solution indeed exists and is unique.

**Hint:** Use a contraction mapping type argument, applied to a subset of \( C[-2, 2] \), which comprises all uniformly bounded functions, such as functions satisfying the bound \( |x(t)| \leq \beta \), for some \( \beta \).

18. Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be a transformation defined by \( T(x) = Ax + b \), where \( A \) is an \( n \times n \) matrix with real entries \( a_{ij} \), and \( b \) is a given vector in \( \mathbb{R}^n \), with components \( b_i \)'s.

(a) Under what conditions on the \( a_{ij} \)'s and \( b_i \)'s is \( T \) a contraction when the norm on \( \mathbb{R}^n \) is the Euclidean one, that is \( \|x\| = \sqrt{\sum_{i=1}^{n} x_i^2} \)?

(b) Under what conditions on the \( a_{ij} \)'s and \( b_i \)'s is \( T \) a contraction when the norm on \( \mathbb{R}^n \) is the maximum norm, that is \( \|x\| = \max_i |x_i| \)? Are these conditions more or less restrictive than the ones obtained in part (a)?
(c) We now wish to compute a fixed point of $T$ by using the iteration (successive approximation)
\[ x_{(i+1)} = Ax_{(i)} + b, \quad i = 0, 1, \ldots \]
where $x_{(0)}$ is an arbitrary starting point. Based on the results you obtained in parts (a) and (b) above, what can you deduce as the conditions on the $a_{ij}$’s and $b$’s for this sequence to converge to a fixed point of $T$.

(d) Show that the condition you obtained in part (c) above holds for the special 3-dimensional case when
\[
A = \begin{pmatrix}
0.5 & 0.2 & 0.2 \\
0.6 & 0.2 & 0.1 \\
0.2 & 0.7 & 0
\end{pmatrix}; \quad b = \begin{pmatrix}
1 \\
2 \\
3
\end{pmatrix}
\]
and obtain the fixed point of $T$ numerically (to 5 significant places) by starting the iteration at (i) the origin (i.e., $x_{(0)} = 0$), and (ii) $x_{(0)} = (-1 \ 1 \ -1)'$.

19. Consider the following integral equation where $\lambda$ is a constant parameter:
\[ x(t) + \lambda \int_0^2 (t - s) x(s) ds = t^3, \quad 0 \leq t \leq 2 \]

(a) If the integral equation is defined on $L_2[0, 2]$, for what values of $\lambda$ does it admit a unique solution?

(b) Is the solution continuous, i.e., does it belong to $C[0, 2]$?

(c) If the integral equation is instead defined on $L_1[0, 2]$, for what values of $\lambda$ does it admit a unique solution using Banach’s contraction mapping theorem? Would this solution be different than the one in (a)?

20. Consider the following linear equation defined on $\ell_1$:
\[
\lambda \sum_{m=1}^{\infty} a_{nm} x_m = 3^{-2n} + x_n, \quad n = 1, 2, \ldots
\]
where $\lambda$ is again a parameter, and $a_{nm}$’s are real numbers.

(a) Under what conditions on $\lambda$ and the $a_{nm}$’s does this equation admit a unique solution $x^* \in \ell_1$? [Use the contraction mapping theorem to solve this problem.]

(b) Now let
\[
a_{nm} = \begin{cases}
\frac{1}{2n^2} & m = n \\
\frac{1}{4m^2n^2} & m \neq n
\end{cases}
\]
Using the result of part (a) above, obtain the conditions on \( \lambda \) such that the map \( T : \ell_1 \to \ell_1 \), defined by

\[
T(x)_n := \lambda \sum_{m=1}^{\infty} a_{nm} x_m - 3^{-2n}
\]

is (i) contraction, (ii) nonexpansive.

21. Let \( X \) be a Banach space with norm \( \| \cdot \| \), and let \( S \) be a compact subset of \( X \). Let \( T \) be a contractive \((\text{not contraction})\) mapping of \( S \) into itself. Show that \( T \) has a unique fixed point, and it can be obtained using the iteration

\[
x_{(n+1)} = T(x_{(n)}), \quad n = 0, 1, \ldots
\]

with an arbitrary starting point \( x_{(0)} \in S \).

The next two problems are on games, and application of fixed point theorems in that context.

22. Consider the zero-sum game with objective function

\[
F : [0, 1] \times [0, 1] \to \mathbb{R}: \quad F(x, y) = -2x^2 + y^2 + 3xy - x - 2y
\]

which is to be maximized with respect to \( x \in [0, 1] \) and minimized with respect to \( y \in [0, 1] \).

(a) Is \( F \) convex-concave? Does the game admit a saddle point? Justify your answers.

(b) Obtain the saddle-point solution (if there exists one). Is it unique?

(c) Consider the sequence \( \{x_k, y_k\} \) generated by:

\[
F(x_{k+1}, y_k) = \max_{x \in [0, 1]} F(x, y_k); \quad F(x_{k+1}, y_{k+1}) = \min_{y \in [0, 1]} F(x_{k+1}, y); \quad k = 0, 1, \ldots
\]

where \( y_0 \in [0, 1] \) is arbitrary. Does this sequence converge, and if it does is the limit the saddle-point solution (if one exists)?

23. Fan’s fixed-point theorem is also useful in the proof of existence of a noncooperative equilibrium \( (\text{Nash equilibrium}) \) in nonzero-sum games. Nonzero-sum games could in fact involve more than two players, but here let us assume that there are actually two players. These are games where the players have different objective functionals, which do not add up to zero as in zero-sum games—thus the name nonzero-sum).

Let \( (X, \rho_X) \) and \( (Y, \rho_Y) \) be two metric spaces, and \( K_X \subset X \) and \( K_Y \subset Y \) be nonempty convex compact subsets of \( X \) and \( Y \), respectively. Player 1 seeks to minimize \( F_1(x, y) \) with respect to \( x \in K_X \), and Player 2 seeks to minimize \( F_2(x, y) \) with respect to \( y \in K_Y \). We say that a pair \( (x^* \in K_X, y^* \in K_Y) \) constitutes a Nash equilibrium, if:

\[
\min_{x \in K_X} F_1(x, y^*) = F_1(x^*, y^*) \quad \text{and} \quad \min_{y \in K_Y} F_2(x^*, y) = F_2(x^*, y^*)
\]
(a) Prove that a two-player nonzero-sum game as formulated above admits a Nash equilibrium if

i) $F_1$ and $F_2$ are both continuous on $X \times Y$; and

ii) $F_1$ is convex in $x \in K_X$ for each $y \in K_Y$, $F_2$ is convex in $y \in Y$ for each $x \in X$.

(b) Obtain the set of all Nash equilibria for the nonzero-sum game where

\[ X = Y = \mathbb{R}, K_X = K_Y = [0, 1]; \quad F_1(x, y) = (y + 1)x^2 - 2x, \quad F_2(x, y) = 2xy - y \]