Problem 1

(30 pts.) Consider the following two definitions of log-space counting problems. A function \( f : \{0, 1\}^* \to \mathbb{N} \) is in \#L_1 if there is a non-deterministic Turing machine \( M_f \) that on input \( x \) of length \( n \) uses \( O(\log n) \) space and is such that the number of accepting paths of \( M_f(x) \) equals \( f(x) \). A function \( f : \{0, 1\}^* \to \mathbb{N} \) is in \#L_2 if there is a relation \( R(.,.) \) that is decidable in log-space and a polynomial \( p \) such that if \( R(x,y) \) then \( |y| \leq p(|x|) \) and such that \( f(x) \) equals \( |y : R(x,y)| \). Prove that all functions in \#L_1 can be computed in polynomial time, while \#L_2 equals \#P.

Problem 2

(30 pts.) Alice and Bob share an arbitrarily long common string \( S \). Alice is given as input a random bit \( x_A \) and Bob a random bit \( x_B \). Without communicating with each other, Alice and Bob wish to output bits \( a \) and \( b \) respectively such that \( x_A \land x_B = a \oplus b \). Prove that any protocol that Alice and Bob follow has success probability at most \( 3/4 \).

Problem 3

Recall that if \( G \) is a \( d \)-regular graph with transition matrix \( M \), then \( G^k \) is the \( d^k \) -regular graph with transition matrix \( M^k \) that has one edge for each path of length \( k \) in \( G \) (with repetitions).

• (30 pts.) Prove that if \( h(G) \geq \epsilon \), then there is a \( k = k(\epsilon) \) that depends only on \( \epsilon \) and not on the size of \( G \) such that \( h(G^k) \geq 1/10 \).

• (10 pts.) Provide a counterexample to the following statement:

\[
 h(G^2) \geq \min\{1/10, 1.01 \times h(G)\}
\]

[Note: the statement may be true (its an open question) if \( G^2 \) is replaced by \( G^3 \).]