Problem 1

(30 pts.) Prove that for every AM[2] protocol for a language L, if the prover and the verifier repeat the protocol k times in parallel (verifier runs k independent random strings for each message) and the verifier accepts if all k copies accept, then the probability that the verifier accepts $x \notin L$ is at most $(1/3)^k$. Note that you cannot assume that the prover is acting independently in each execution. (Use definition 8.6 for IP from Arora Barak).

Problem 2

(30 pts.) Define a language $L$ to be downward-self-reducible if there is a polynomial time algorithm $R$ that for any $n$ and $x \in \{0,1\}^n$, $R^{L(n-1)}(x) = L(x)$ where by $L_k$ we denote an oracle that solves $L$ on inputs of size at most $k$. Prove that if $L$ is downward-self-reducible then $L \in \text{PSPACE}$.

Problem 3

Recall that the trace of a matrix $A$, denoted $tr(A)$ is the sum of the entries along its diagonal.

- (10 pts.) Prove that if an $n \times n$ matrix $A$ has eigenvalues $\lambda_1, \ldots, \lambda_n$, then $tr(A) = \sum_{i=1}^{n} \lambda_i$.

- (10 pts.) Prove that if $A$ is a random walk matrix of an $n$-vertex graph $G$ and $k \geq 1$, then $tr(A^k)$ is equal to $n$ times the probability that if we select a vertex $i$ uniformly at random and take a $k$ step random walk from $i$, then we end up back in $i$.

- (10 pts.) Prove that for every $d$-regular graph $G$, $k \in \mathbb{N}$ and vertex $i$ of $G$, the probability that a path of length $k$ from $i$ ends up back in $i$ is at least as large as the corresponding probability in $T_d$, where $T_d$ is the complete $(d-1)$-ary tree.
of depth $k$ rooted at $i$. (that is, every internal vertex has degree $d$, one parent and $d−1$ children.)

• (10 pts.) Prove that for even $k$, the probability that a path of length $k$ from the root $v$ of $T_d$ ends up back at $v$ is at least $2^{k−k\log(d−1)/2+o(k)}$. 