1. We say a discrete memoryless channel is symmetric if the set of outputs can be partitioned into subsets in such a way that for each subset the matrix of transition probabilities has the property that each row (if more than 1) is a permutation of each other row and each column is a permutation of each other column.

(a) Prove that for a symmetric channel with any number of inputs, the uniform distribution over the inputs is the optimal one, in terms of maximizing the mutual information over the channel.

(b) Are there channels that are not symmetric whose optimal input distributions are uniform? Find one, or prove that there are none.

2. Prove that no output $y$ is unused by an optimal (capacity-achieving) input distribution, unless it is unreachable, that is, the channel transition probabilities satisfy $P(y|x) = 0$ for all $x$.

3. Consider two discrete memoryless channels such that the output domains of the two channels are non-overlapping. At each time, you are allowed to use either of the two channels, but not both. Show that the capacity $C$ of the above channel model is given by $2C = 2C_1 + 2C_2$ where $C_1, C_2$ are the capacities of the two channels under consideration.

4. The Z-channel on binary inputs and outputs is the following: $P(Y = 0|X = 0) = 1$ and $P(Y = 0|X = 1) = 0.5$ and $P(Y = 1|X = 1) = 0.5$. Compute the capacity of the Z channel.

5. Consider a channel with binary inputs and ternary outputs that has both erasures and errors: $P(Y = i|X = i) = 1 - \alpha - \epsilon$ and $P(Y = i|X \neq i) = \epsilon$ for $i \in \{0, 1\}$. Further, $P(Y = \epsilon|X = i) = \alpha$ for $i \in \{0, 1\}$. Find an analytic expression for the capacity of the channel and verify your answer by specializing to the special case of $\alpha = 0$ and $\epsilon = 0$.

6. Consider the memoryless binary channel given by $Y_i = X_i \oplus Z_i$, where $X_i, Y_i, Z_i$ all take values in 0, 1, and $\oplus$ denotes addition modulo-2. There are channel states $S_i$ which determine the noise level of $Z_i$ as follows: $S_i$ is binary valued, taking values in the set $G, B$, with $P(S_i = G) = \frac{2}{3}$ and $P(S_i = B) = \frac{1}{3}$. The conditional distribution $P(Z_i = 1|S_i = s) = \frac{1}{4}$ if $s = G$ and $P(Z_i = 1|S_i = s) = \frac{1}{3}$ if $s = B$. The channel and state are memoryless: $(S_i, Z_i)$ are i.i.d. (in pairs), independent of the channel input sequence $X_i$. 


(a) What is the capacity of this channel when both the encoder and the decoder have access to the state sequence \( \{S_i\}_{i \geq 1} \)?

(b) What is the capacity of this channel when neither the encoder nor the decoder have access to the state sequence?

(c) What is the capacity of this channel when only the decoder knows the state sequence?