ECE 563 Spring 2016 **Homework 4**

Date Assigned: 4 October 2016.
Date Due: 13 October 2016 in class.

1. A channel with input alphabet 0,1,2,3,4 has transition probability of the form $p(y|x) = 0.5$ if $y = x \pm 1 \text{mod} 5$, and zero, otherwise.

   (a) Compute the capacity of this channel.

   (b) Now we try to communicate over this channel with zero error probability. Clearly the rate of communication is at least 1 bit. This can be achieved in a single channel use (block length unity) by either sending input 0 or 1 with equal probability. By considering codes of block length 2, show that 5 codewords can be sent with zero error probability – leading to a data rate of $0.5 \log_2 5$.

   Notes: In a remarkable paper in 1979, Lovász showed that $0.5 \log_2 5$ is actually the zero-error capacity of this so-called "pentagon" channel. This paper was remarkable for many reasons, among which was the invention of the now-classical mathematical programming approach of semi-definite relaxation to a combinatorial problem. A reading exercise is to enjoy this brilliant and seminal paper by reading through it carefully and constructing for yourself a fully self contained proof of the zero-error capacity of the pentagon channel.

   (c) Again, we try to communicate over this channel with zero error probability, but this time around we have lossless instantaneous feedback from the receiver. Thus, the next channel use can be predicated based on this feedback. Construct a simple scheme that has a data rate of $\log_2 2.5$.

   Notes: This rate was shown to be optimal for zero error communication with feedback by Shannon in another classic paper in 1956 (doi:10.1109/TIT.1956.1056798, for those who want to look it up).

2. Consider a commander of an army beseiged in a fort for whom the only means of communication to his allies is a set of carrier pigeons. Assume that each carrier pigeon can carry one byte (8 bits), that pigeons are released once every 5 minutes and that each pigeon takes exactly 3 minutes to reach its destination.

   (a) Assuming the pigeons reach safely, what is the capacity of the link in bits/hour?

   (b) Now suppose that the enemies try to shoot down the pigeons and that they manage to hit a fraction α of them. Since the pigeons are sent at a constant rate, the receiver knows which pigeons are missing. What is the capacity of this link?

(c) Now assume that the enemy is more cunning and that every time they shoot down a pigeon they send out a dummy pigeon carrying a random letter (chosen uniformly among all possible 8-bit length sequences). What is the capacity of this link in bits/hour?

3. Suppose that the average height of kids in a room is 5 ft. Suppose the average height is 100 lb.

   (a) Argue that no more than one-third of the population is 15 ft tall.

   (b) Find an upper bound on the fraction of 300 lb, 10 footers in the room.

4. Consider the memoryless "binary multiplicative" channel: $y = xz$ where z is Beroulli(q) and independent of the binary input x.

   (a) Find the capacity of this channel.

   (b) Suppose the receiver also has access to z along with the output y. Find the capacity of this channel.

   (c) Now suppose that the "noise" z is actually a signal being sent by another user (who has no coordination with the regular transmitter, and is hence independent of x). This time there is no constraint on q. Both these users are trying to communicate through the binary multiplicative channel to the (common) destination.

      i. Argue, using a very simple pre-arranged coding scheme, that both users can communicate reliably at a rate of 0.5 bits per cannel use.

      ii. Bonus: Shannon showed that both users can communicate reliably at a rate of 0.617 bits per cannel use, if they have feedback about the output of the multiplicative channel. Reconstruct this argument and proof for yourself. Hint: The coding scheme is simple and non-interactive. In a very nice paper in 1982, Schalkwijk showed a very nice interactive scheme that lets both users communicate reliably at a rate of 0.619 bits per cannel use. Going through this (short) paper is a reading exercise for the more- interested student.

5. We say a discrete memoryless channel is symmetric if the set of outputs can be partitioned into subsets in such a way that for each subset the matrix of transition probabilities has the property that each row (if more than 1) is a permutation of each other row and each column is a permutation of each other column.

   (a) Prove that for a symmetric channel with any number of inputs, the uniform distribution over the inputs is the optimal one, in terms of maximizing the mutual information over the channel.

   (b) Are there channels that are not symmetric whose optimal input distributions are uniform? Find one, or prove that there are none.

6. Prove that no output y is unused by an optimal (capacity-achieving) input distribution, unless it is unreachable, that is, the channel transition probabilities satisfy $P(y|x) = 0$ for all x.

7. Consider two discrete memoryless channels such that the output domains of the two channels are non-overlapping. At each time, you are allowed to use either of the two channels, but not both. Show that the capacity C of the above channel model is given by $2^C = 2^{C_1} + 2^{C_2}$ where C_1, C_2 are the capacities of the two channels under consideration.

8. The Z-channel on binary inputs and outputs is the following: $P(Y = 0|X = 0) = 1$ and$P(Y = 0|X = 1) = 0.5$ and $P(Y = 1|X = 1) = 0.5$. Compute the capacity of the Z channel.

9. Consider a channel with binary inputs and ternary outputs that has both erasures and errors: $P(Y = i|X = i) = 1 - \alpha - \epsilon$ and $P(Y \neq i|X = i) = \epsilon$ for $i \in \{0, 1\}$. Further, $P(Y = e|X = i) = \alpha$ for $i \in \{0, 1\}$. Find an analytic expression for the capacity of the channel and verify your answer by specializing to the special case of $\alpha = \epsilon = 0$.