ECE 563 Spring 2016 Homework 2

Date Assigned: 06 September 2016.
Date Due: 15 September 2016 in class.

1. Consider the random variable X that takes values $x_1, \dots x_7$ with probabilities $p_i = P(X = x_i)$ as follows: $p_1 = 049, p_2 = 0.26, p_3 = 0.12, p_4 = 0.04, p_5 = 0.04, p_6 = 0.03, p_7 = 0.02$.
   (a) Find a binary Huffman code for X.
   (b) Find the expected code length for this encoding.
   (c) Find a ternary Huffman code for X.

2. In lecture we found that the optimal average codeword length is less than $H + 1$ where H is the entropy of the source. Give an example of a random variable whose expected length of the optimal code is arbitrarily close to $H + 1$. In other words, for any fixed $\epsilon > 0$ construct a source probability distribution for which the optimal code has average length $L > H + 1 - \epsilon$.

3. Player A chooses some object in the universe, and player B attempts to identify the object with a series of yes-no questions. Suppose that player B is clever enough to use the code achieving the minimal expected length with respect to player A?s distribution. We observe that player B requires an average of 38.5 questions to determine the object. Find a rough lower bound to the number of objects in the universe.

4. A code is not uniquely decodable if and only if there exists a finite sequence of code symbols which can be resolved into sequences of codewords in two different ways; for example the codeword sequence could be split as either $(A_1 A_2 A_3 \dots A_m)$ or $(B_1 B_2 \dots B_n)$ where each A_i and B_j is a codeword. If the length of A_1 is larger than that of B_1 (without loss of generality), then B_1 must be a prefix of A_1 with some "dangling suffix" left over in A_1. Each dangling suffix must in turn be either a prefix of a codeword (in this case the dangling suffix of A_1 is a prefix of B_2, assuming that the length of $B_1 B_2$ is more than the length of A_1) or have another codeword as its prefix, resulting in another dangling suffix. Finally, the last dangling suffix in the sequence must also be a codeword. Thus one can set up a test for unique decidability – this is the so-called Sardinas-Patterson test – in the following way. Construct a set S of all possible dangling suffixes. The code is uniquely decodable if and only if S contains no codeword.
   (a) State the precise rules for building the set S.

(b) Suppose the codeword lengths are $l_i, i = 1 \dots m$. Find a good upper bound on the number of elements in the set S.

(c) Determine which of the following codes is uniquely decodable.

i. 0,10.11

ii. 0,01,11

iii. 0,01,10

iv. 0,01

v. 00,01,10,11

vi. 110,11,10

vii. 110,11,100,00,10

5. Consider the following method of generating a code for a random variable X that takes on m values $1, 2, \dots, m$ with probabilities $p_1, \dots, p_m$. Also suppose that the probabilities are ordered, so $p_1 \geq p_2 \geq \dots \geq p_m$. Define $F_i = \sum_{\ell=1}^{i-1} p_\ell$, the sum of the probabilities of symbols less than i. Then the codeword for i is the number $F_i \in [0, 1]$ rounded off to ℓ_i bits, where $\ell_i = \lceil \log \frac{1}{p_i} \rceil$.

(a) Show that the code constructed by this process is prefix-free and that the average length L satisfies $H(X) \leq L \leq H(X) + 1$.

(b) Construct the code for the probability distribution $(0.5, 0.25.0.125, 0.125)$.

6. One is given six bottles of wine. It is known that precisely one bottle has gone bad (tastes terrible). From inspection of the bottles it is determined that the probability p_i of the ith bottle is bad is given by: $p_1 = \frac{8}{23}$, $p_2 = \frac{6}{23}$, $p_3 = \frac{4}{23}$,$p_4 = \frac{2}{23}$,$p_5 = \frac{2}{23}$,$p_6 = \frac{1}{23}$. Tasting will determine the bad wine. Suppose you taste one wine at a time. Choose the order of tasting to minimize the expected number of tastings required to determine the bad bottle. Remember, if the first five wines pass the test then you dont need to taste the last.

(a) What is the expected number of tastings required?

(b) Which bottle should be tasted first?

(c) Now you get smart. For the first sample, you mix some of the wines in a fresh glass and sample the mixture. You proceed, mixing and tasting, stopping when the bad bottle has been determined. Now answer this question and the next in this new framework.

What is the minimum expected number of tastings required to determine the bad bottle?

(d) What mixture should be tasted first?

7. Which of the following codeword lengths can be the word lengths of a 3-ary Huffman code, and which cannot?

(a) (1,2,2,2,2).

(b) (2,2,2,2,2,2,2,2,3,3,3).

8. We have $n_{ij} = n_{ji}$ cards painted with color i (chosen from a palette of M colors) on one side and with color j on the other. The cards are shuffled and we are allowed to see one side of the card at the top of the deck. Find the entropy of the color of the other side. Do not assume that $n_{ii} = 0$.

9. **Optional**: Suppose X is a Binomial random variable with parameters (n, p).

   (a) Show that $H(X) = nh_2(p) + \mathbb{E}\begin{pmatrix} n \\ X \end{pmatrix} = \frac{1}{2}\log(2\pi ep(1-p)n) + O(\frac{1}{n})$.

   (b) Show that
   $$\frac{1}{2}\log\frac{\pi n}{2} \leq H(X) \leq \frac{1}{2}\log \pi e\lceil\frac{\pi n}{2}\rceil.$$
   Hint: Consider looking at S.-C. Chang and J. E. J. Weldon, "Coding for t-user multiple-access channels," *IEEE Trans. Information Theory*, vol. 25, no. 6, pp. 684?691, Nov. 1979.