1. [Performance of MPSK]

(a) Using the Intelligent Union Bound, show that the symbol error probability for MPSK signaling in AWGN is bounded by

\[ P_e \leq 2Q\left( \sqrt{\frac{2E_s}{N_0}} \sin \frac{\pi}{M} \right). \]

Solution:

We only need the error regions caused by adjacent points for the intelligent union bound so

\[ P_e \leq 2 Q \left( \sqrt{\frac{d_{\text{min}}^2}{2N_0}} \right). \]

For PSK

\[ d_{\text{min}}^2 = 2E_s \left( 1 - \cos \frac{2\pi}{M} \right) \]

and therefore

\[ P_e \leq 2 Q \left( \sqrt{\frac{E_s (1 - \cos \frac{2\pi}{M})}{N_0}} \right). \]
Using $\cos 2x = 1 - 2\sin^2 x$, we get

$$P_e \leq 2Q \left( \sqrt{\frac{\mathcal{E}_s (1 - \cos \frac{2\pi}{M})}{N_0}} \right) = 2Q \left( \sqrt{\frac{2\mathcal{E}_s \sin^2 \frac{\pi}{M}}{N_0}} \right) = 2Q \left( \frac{2\sqrt{\mathcal{E}_s}}{N_0} \sin \frac{\pi}{M} \right)$$

(b) Now, derive the following exact expression for $P_e$.

$$P_e = \frac{1}{\pi} \int_{0}^{(M-1)\pi/M} \exp \left[ -\frac{\mathcal{E}_s \sin^2 (\pi/M)}{N_0 \sin^2 \theta} \right] d\theta .$$

**Hint:** One way to proceed is to shift the origin to the signal point under consideration and use polar co-ordinates with the appropriate limits of integration.

**Solution:**

![Figure 2: Region of Integration for $P_e$ is everything outside the cone ($R(\theta) = b(\theta)$ below)](image2)

![Figure 3: Triangle used to find $b(\theta)$](image3)

By symmetry it is sufficient to calculate the probability of error for any one point as all the others will be the same. Following the suggestion in the problem we will move one point to the origin and integrate over the error region in polar coordinates. Denoting the Cartesian coordinates by $x_I$ and $x_Q$, we see that the polar coordinates are given in terms of $x_I$ and $x_Q$ as

$$\rho = \sqrt{x_I^2 + x_Q^2}, \quad \theta = \tan^{-1}(x_Q/x_I), \quad \text{equivalently} \quad x_I = \rho \cos \theta, \quad x_Q = \rho \sin \theta$$

First, we need to describe our region of integration in polar coordinates. It is sufficient to consider the upper half of the region as we can just integrate over it and multiply by 2 by symmetry. We let $\theta$ range from 0 to $(M-1)\pi/M$ and $\rho$ range from $b(\theta)$ to $\infty$. To describe the region we need to find $b(\theta)$. To do this consider the triangle and apply the law of sines to get

$$\frac{b(\theta)}{\sin \frac{\pi}{M}} = \frac{\sqrt{\mathcal{E}_s}}{\sin (\pi - \frac{\pi}{M} - \theta)} \Rightarrow b(\theta) = \sqrt{\mathcal{E}_s} \sin \frac{\pi}{M} \sin (\pi - \frac{\pi}{M} - \theta)$$

Since we moved the signal point to the origin, the error probability is given by integrating a
Consider the following two bit assignments for QPSK

\[ P_e = \int \int_{\text{Error region}} \frac{1}{\pi N_o} \exp \left\{ -\frac{x_1^2 + x_2^2}{N_o} \right\} \, dx_1 \, dx_Q = 2 \int_{\theta=0}^{\frac{\pi}{M-1}} \int_{\rho=b(\theta)}^{\infty} \frac{1}{\pi N_o} \exp \left\{ -\frac{\rho^2}{N_o} \right\} \, \rho \, d\rho \, d\theta \]

\[ = \int_{\theta=0}^{\frac{\pi}{M-1}} \int_{\rho=b(\theta)}^{\infty} 2\rho \frac{1}{\pi N_o} \exp \left\{ -\frac{\rho^2}{N_o} \right\} \, d\rho \, d\theta = \frac{1}{\pi} \int_{\theta=0}^{\frac{\pi}{M-1}} \frac{1}{\pi N_o} \exp \left\{ -\frac{\rho^2}{N_o} \right\} \, d\theta \]

\[ = \frac{1}{\pi} \int_{\theta=0}^{\frac{\pi}{M-1}} \exp \left\{ -\frac{\rho^2}{N_o} \right\} \, d\theta \]

2. [Gray coding for QPSK]

Consider the following two bit assignments for QPSK

\[ \begin{array}{c|c|c|c|c|c} 
  & 01 & 00 & 01 & 00 & 10 & 11 \\
 11 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 01 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 10 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 00 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
 \end{array} \]

(a) Show that assignment (1), which corresponds to Gray coding, results in an average bit error probability of \( P_b = Q(\sqrt{2\gamma_b}) \).

**Solution:** We did this in class. For assignment (1), it is easy to see by symmetry that the bit error probability is the same for both the left bit and the right bit and for each of the four symbols that is sent. Thus \( P_b \) can be computed by considering any symbol (say 00) and looking at the error probability for the first bit. When 00 is sent, it is clear that the error region is the lower half plane. Thus

\[ P_b = P\{\sqrt{E_s/2} + W_Q \leq 0\} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\gamma_b}} = Q(\sqrt{\gamma_b}) = Q(\sqrt{2\gamma_b}) \]

(b) Show that under assignment (2), the first bit (from the left) sees an average probability of error of \( Q(\sqrt{2\gamma_b}) \), whereas the second bit sees an average probability of error of \( 2Q(\sqrt{2\gamma_b})[1 - Q(\sqrt{2\gamma_b})] \). Thus

\[ P_b = \frac{1}{2}Q(\sqrt{2\gamma_b}) + Q(\sqrt{2\gamma_b})[1 - Q(\sqrt{2\gamma_b})] = \frac{1}{2}Q(\sqrt{2\gamma_b}) \]

**Solution:** For assignment (2), the first bit sees the same error patterns as in assignment (1), and therefore \( P_{b,1} = Q(\sqrt{2\gamma_b}) \). But the second bit sees a different error pattern. In particular, when 00 is sent, the error region is the second and fourth quadrants. By symmetry, the error probability for each of these two regions is the same. Thus,

\[ P_{b,2} = 2P\{\sqrt{E_s/2} + W_Q \geq 0\} \cap \{\sqrt{E_s/2} + W_I \leq 0\} \]

\[ = 2Q\left(\sqrt{\frac{E_s}{N_0}}\right) \left[1 - Q\left(\sqrt{\frac{E_s}{N_0}}\right)\right] = 2Q(\sqrt{2\gamma_b})[1 - Q(\sqrt{2\gamma_b})] \]

and \( P_b = \frac{1}{2}[P_{b,1} + P_{b,2}] \).
3. **[NNA versus IUB]**

Consider the 8-ary constellation shown below:

(a) Assuming equally likely symbols, carefully draw the MPE decision regions for this constellation

**Solution:** See figure below

(b) Find the NNA for $P_e$ in terms of $\gamma_s$.

**Solution:** First note that $E_s = d^2[2 \times \frac{1}{4} + 2 \times \frac{9}{4} + 4 \times \frac{5}{4}] / 8 = \frac{5}{4} d^2$. Thus $\gamma_s = \frac{5}{4} d^2$.

Now points 1 and 2 have 4 NNs, points 3 and 4 have 1 NN, and the remaining points have 2 NNs. Thus the NNA for $P_e$ is given by:

$$P_e \approx \frac{1}{8} Q \left( \sqrt{\frac{d^2}{2 N_0}} \right) [2 \times 4 + 2 \times 1 + 4 \times 2] = \frac{9}{4} Q \left( \sqrt{\frac{d^2}{2 N_0}} \right) = \frac{9}{4} Q \left( \sqrt{\frac{2 \gamma_s}{5}} \right)$$

(c) Find the IUB for $P_e$ in terms of $\gamma_s$.

**Solution:** For points 1 and 2, the NNs cover the error region. For points 3 and 4, we need to include two more points at distance $\sqrt{2} d$ to cover the error region. And for points 5, 6, 7, and 8, we need to include one more point at distance $\sqrt{2} d$ to cover the error region. Thus the IUB is given by:

$$P_e \leq \frac{1}{8} Q \left( \sqrt{\frac{d^2}{2 N_0}} \right) [2 \times 4 + 2 \times 1 + 4 \times 2] + \frac{1}{8} Q \left( \sqrt{\frac{d^2}{N_0}} \right) [2 \times 2 + 4 \times 1]$$

$$= \frac{9}{4} Q \left( \sqrt{\frac{2 \gamma_s}{5}} \right) + Q \left( \sqrt{\frac{4 \gamma_s}{5}} \right)$$

(d) Find an exact expression for $\max_m P_{e,m}$, the largest conditional symbol error probability.
**Solution:** It is clear from the figure that \( \max_m P_{e,m} = P_{e,1} = P_{e,2} \). By shifting the constellation point to the origin, we have

\[
P_{c,1} = \left[ 1 - 2Q \left( \frac{d/2}{\sqrt{N_0}/2} \right) \right]^2 = \left[ 1 - 2Q \left( \frac{2\gamma_b}{\sqrt{5}} \right) \right]^2
\]

and \( P_{e,1} = 1 - P_{c,1} \).

4. **[Gray Coding and Bit Error Probability]**

Consider the 8-ary QAM constellation shown below (where all nearest neighbors are equidistant):

(a) Determine whether you can label the signal points using three bits so that nearest neighbors differ by at most one bit (Gray coding). If so, find such a labeling. If not, state why not and find a labeling that minimizes the maximum number of bit transitions between neighbors.

**Solution:** The Gray coding bit assignment is shown in the figure below.

(b) For the labelings found in part (a), compute the nearest neighbor approximation for the average bit error probability \( P_b \) as a function of \( d^2/N_0 \).

**Solution:**

\[
P_{b,1} \approx \frac{1}{8} Q \left( \sqrt{\frac{d^2}{2N_0}} \right) [1 + 1 + 1 + 1 + 1 + 1 + 1 + 1] = Q \left( \sqrt{\frac{d^2}{2N_0}} \right)
\]

\[
P_{b,2} \approx \frac{1}{8} Q \left( \sqrt{\frac{d^2}{2N_0}} \right) [1 + 1 + 1 + 0 + 0 + 0 + 0 + 0] = \frac{1}{2} Q \left( \sqrt{\frac{d^2}{2N_0}} \right)
\]

\[
P_{b,3} \approx \frac{1}{8} Q \left( \sqrt{\frac{d^2}{2N_0}} \right) [1 + 1 + 1 + 0 + 0 + 0 + 0 + 0] = \frac{1}{2} Q \left( \sqrt{\frac{d^2}{2N_0}} \right)
\]

Thus \( P_b \approx \frac{2}{3} Q \left( \sqrt{\frac{d^2}{2N_0}} \right) \), which is as it should be for Gray coding (see notes 8).
5. ["Semi-Orthogonal" Signal Set]
Consider the signal set with $M = 2N$ signals given by:

$$s_m(t) = \begin{cases} \sqrt{E}g_m(t) & m = 0, 1, \ldots, N - 1 \\ j\sqrt{E}g_{m-N}(t) & m = N, \ldots, M - 1. \end{cases}$$

where $\{g_k(t)\}_{k=1}^N$ are real-valued orthonormal functions. Clearly this signal set satisfies: $\Re[\rho_{k,\ell}] = 0$, for $k \neq \ell$, but not $\rho_{k,\ell} = 0$, for $k \neq \ell$

(a) Argue that $R_k = \langle r(t), g_k(t) \rangle$, $k = 0, 1, \ldots, N - 1$, form sufficient statistics for optimal decision making at the receiver for an AWGN channel. 

**Solution:** The functions $\{g_k(t)\}$ form a basis for our complex signal space. By the principle of irrelevance, we get a sufficient statistics if we project our received signal onto each of $g_k$’s.

(b) Now define the $M$ real-valued statistics

$$y_m = \begin{cases} r_{m,I} & m = 0, 1, \ldots, N - 1 \\ r_{(m-N),Q} & m = N, \ldots, M - 1. \end{cases}$$

Show that the MPE decision rule is given by

$$\hat{m}_{\text{MPE}} = \arg\max_m y_m$$

**Solution:** Assume equal priors so the MPE detector is the ML detector. Let $W \sim \mathcal{CN}(0, N_o I)$. Then for $0 \leq m \leq N - 1$:

$$R = \begin{bmatrix} 0 & \cdots & 0 & \sqrt{E} & 0 & \cdots & 0 \end{bmatrix}^\top + W$$

and for $N \leq m \leq 2N - 1$:

$$R = \begin{bmatrix} 0 & \cdots & 0 & j\sqrt{E} & 0 & \cdots & 0 \end{bmatrix}^\top + W$$

where the non-zero components in both vectors are in the $m$-th positions. The conditional pdf for $0 \leq m \leq N - 1$ is given by

$$p_m(z) = \frac{1}{(\pi N_o)^N} \exp \left\{ -\frac{\|z - y_m\|^2}{N_o} \right\} = \frac{1}{(\pi N_o)^N} \exp \left\{ -\frac{(r_{m,I} - \sqrt{E})^2 + \cdots}{N_o} \right\}$$

$$= \frac{1}{(\pi N_o)^N} \exp \left\{ 2\sqrt{E} y_m - \frac{\|z - \mathcal{N}(0, N_o)\|^2}{N_o} \right\}$$

Similarly for $N \leq m \leq M - 1$:

$$p_m(z) = \frac{1}{(\pi N_o)^N} \exp \left\{ 2\sqrt{E} y_m - \frac{\|z - \mathcal{N}(0, N_o)\|^2}{N_o} \right\}$$

Only the $y_m$ term is a function of the message and the exponential is monotonic increasing, so the MPE detector is given by

$$\hat{m}_{\text{MPE}} = \arg\max_{m \in \{0, 1, \ldots, M-1\}} y_m$$

(c) Find an expression for $P_e$ for the MPE decision rule.

**Solution:** By symmetry $P_e = P_{e,0}$, so we only need to calculate the probability of error given that we sent 0. Given that we sent 0, $Y_0 = \sqrt{E} + W_{1, I}$ and $Y_k = W_{k, I}$ or $Y_k = W_{k-M, Q}$ for $k > 0$. Thus $Y_k$ is a set of i.i.d. $\mathcal{N}(0, N_o/2)$ random variables for $k \neq 0$, and $W_{0, I}$ is independent of them.
Therefore the probability of correct decision making when 1 is sent is the same as in the case of coherent demodulation in the completely orthogonal case (see notes 9).

\[
P_{c,0} = \frac{1}{\sqrt{\pi N_0}} \int_{-\infty}^{\infty} \left[ 1 - Q \left( \frac{\sqrt{E} + x}{\sqrt{N_0}/2} \right) \right]^{M-1} e^{-\frac{x^2}{2N_0}} dx
\]

\[
= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [1 - Q(t)]^{M-1} e^{-\frac{1}{2} (t - \sqrt{E})^2} dt
\]

where \( \gamma_s = \frac{\gamma_b}{N_0} = \frac{\gamma}{N_0} \). And

\[
P_e = 1 - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} [1 - Q(t)]^{M-1} e^{-\frac{1}{2} (t - \sqrt{E})^2} dt
\]

6. **Asymptotic Performance of Orthogonal Signaling**

In this problem you will show the following result for \( M \)-ary orthogonal modulation

\[
\lim_{M \to \infty} P_c = \begin{cases} 
1 & \gamma_b > \ln 2 \\
0 & \gamma_b < \ln 2
\end{cases}
\]

where \( P_c \) is the probability of correct decision making.

Recall that we showed in class that

\[
P_c = \int_{-\infty}^{\infty} \left[ 1 - Q \left( x + \sqrt{2\gamma_b \log_2 M} \right) \right]^{M-1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx
\]

(a) Show that for any \( x \),

\[
\lim_{M \to \infty} \left[ 1 - Q \left( x + \sqrt{2\gamma_b \log_2 M} \right) \right]^{M-1} = \begin{cases} 
1 & \gamma_b > \ln 2 \\
0 & \gamma_b < \ln 2
\end{cases}
\]

Hint: Use L’Hopital’s rule on the log of the expression before taking the limit.

**Solution:** Define \( z(M) = x + \sqrt{2\gamma_b \log_2 M} \). Taking the log gives

\[
\ln \left[ 1 - Q(z(M)) \right]^{M-1} = (M - 1) \ln (1 - Q(z(M)))
\]

\[
= \frac{\ln (1 - Q(z(M)))}{M - 1}
\]

If we let \( M \to \infty \), we get an indeterminate form \( \frac{0}{0} \), so we apply L’Hopital’s rule. First differentiate the denominator

\[
\frac{\partial}{\partial M} \left[ \frac{1}{M - 1} \right] = \frac{-1}{(M - 1)^2}
\]

For \( Z \)

\[
\frac{\partial Z}{\partial M} = \sqrt{\frac{\gamma_b}{2\ln 2}} \frac{1}{M \sqrt{\ln M}}
\]

so the derivative of the numerator is given by

\[
\frac{\partial \text{num}}{\partial M} = \frac{1}{1 - Q(z(M))} \sqrt{2\pi} \exp \left\{ -\frac{x^2}{2} \right\} \frac{\partial Z}{\partial M}
\]

\[
= \frac{1}{1 - Q(z(M))} \sqrt{2\pi} \exp \left\{ -\frac{x^2}{2} \right\} \exp \left\{ -x \sqrt{2\gamma_b \log_2 (M)} \right\} M^{-\frac{\gamma_b}{2\sqrt{\ln M}}}
\]

\[
= \frac{1}{1 - Q(z(M))} \sqrt{\frac{\gamma_b}{4\pi \ln 2}} \exp \left\{ -\frac{x^2}{2} \right\} \exp \left\{ -x \sqrt{2\gamma_b \log_2 (M)} \right\} \cdot M^{-\frac{\gamma_b}{M \sqrt{\ln M}}}
\]

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Then we consider the limits as \( M \to \infty \) of

\[
- \frac{1}{1 - Q(z(M))} \sqrt{\frac{\gamma_b}{4\pi \ln 2}} \exp \left\{ -\frac{x^2}{2} \right\} \exp \left\{ -x \sqrt{2\gamma_b \log_2(M)} \right\} M^{-\frac{\gamma_b}{2}} (M - 1)^2 M^{1 - \frac{\gamma_b}{2}} \frac{1}{\sqrt{\ln M}}
\]

Simplifying we get

\[
- \frac{(1 - \frac{1}{M})^2}{1 - Q(z(M))} \sqrt{\frac{\gamma_b}{4\pi \ln 2}} \exp \left\{ -\frac{x^2}{2} \right\} \exp \left\{ -x \sqrt{2\gamma_b \log_2(M)} \right\} M^{1 - \frac{\gamma_b}{2}} \frac{1}{\sqrt{\ln M}}
\]

Then the exponential term in \( M \) in this expression dominates the growth, so if \( \gamma_b > 1 \), then this quantity approaches \( 0 \) and if \( \gamma_b < 1 \), then this quantity approaches \( -\infty \). Therefore

\[
\lim_{M \to \infty} \left[ 1 - Q(z(M)) \right]^{-1} = \begin{cases} 1 & \text{if } \gamma_b > 1, \\ 0 & \text{if } \gamma_b < 1 \end{cases}
\]

(b) Use the result of part (a) to arrive at the desired result.

**Solution:** Taking the limit inside the integral gives

\[
P_c = \int_0^\infty \lim_{M \to \infty} \left[ 1 - Q(z(M)) \right]^{-1} \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\} dx
\]

\[
= \begin{cases} \int_0^\infty \frac{1}{\sqrt{2\pi}} \exp \left\{ -\frac{x^2}{2} \right\} dx & \text{if } \gamma_b > 1, \\ 0 & \text{if } \gamma_b < 1 \end{cases}
\]

\[
= \begin{cases} 1 & \text{if } \gamma_b > 1, \\ 0 & \text{if } \gamma_b < 1 \end{cases}
\]

7. **[Noncoherent Demodulation of Linearly Modulated Signals]**

The received signal for one symbol period for linear memoryless modulation on an ideal AWGN channel is given by:

\[
r(t) = \sqrt{E_m} e^{j \theta_m} g(t) e^{j \phi} + w(t)
\]

where the phase offset \( \phi \) is due to the delay introduced by channel. If \( \phi \) is known at the receiver, we can correct for it (by projecting \( g(t) \) on \( g(t) e^{j \phi} \) to produce the sufficient statistic) and suffer no loss in detection performance. However, if \( \phi \) is not known, we may project \( r(t) \) on \( g(t) \) to get the sufficient statistic

\[
R = \sqrt{E_m} e^{j \theta_m} e^{j \phi} + W
\]

where \( W \sim \mathcal{CN}(0, N_0) \). Since \( \phi \) is not of direct interest to the receiver, we treat it as a nuisance parameter. As we saw in class, there are two ways to deal with such parameters.

(a) Assume that \( \phi \in [0, 2\pi] \), and find \( \hat{m}_{JML}(r) \) using the joint ML approach. Interpret your answer.

**Solution:** We know that \( R \sim \mathcal{CN}(\sqrt{E_m} e^{j \theta_m} e^{j \phi}, N_0) \), and therefore the conditional pdf of \( R \) when \( m \) is sent is given by

\[
p_{m, \phi}(r) = \frac{1}{\pi N_0} \exp \left\{ -\frac{1}{N_0} \left| r - \sqrt{E_m} e^{j \theta_m} e^{j \phi} \right|^2 \right\}
\]

Then the joint ML estimate is computed using

\[
p_{m}^{(\text{max})}(r) = \max_{\phi} p_{m, \phi}(r) = \max_{\phi} \frac{1}{\pi N_0} \exp \left\{ -\frac{1}{N_0} \left| r - \sqrt{E_m} e^{j \theta_m} e^{j \phi} \right|^2 \right\}
\]

\[
= \frac{1}{\pi N_0} \exp \left\{ -\frac{1}{N_0} \left| r - \sqrt{E_m} \right|^2 \right\}
\]

since for any given \( m \), we can choose \( \phi \) such that \( r \) and \( \sqrt{E_m} e^{j \theta_m} e^{j \phi} \) have the same phase. Therefore

\[
\hat{m}_{JML}(r) = \arg \max_m p_{m}^{(\text{max})}(r) = \arg \min_m \left| r - \sqrt{E_m} \right|^2
\]

It is clear from this expression that we can only distinguish the symbols at the receiver if they have different amplitudes (energies).
(b) Now assume that $\phi$ is a random variable that is uniformly distributed on $[0, 2\pi]$, and find $\hat{m}_{\text{MAP}}(y)$.

Simplify your answer as much as possible (note that your answer can be written in terms of the Bessel function $I_0$).

Note: You should see from this problem that noncoherent demodulation of linearly modulated signals is not a very good idea.

Solution: From the derivations in class and the notes on noncoherent communications,

$$p_m^{(avg)}(r) = \frac{1}{\pi N_o} \exp \left\{ -\frac{1}{N_o}(|r|^2 + \mathcal{E}_m) \right\} I_0 \left( \frac{2|r|\sqrt{\mathcal{E}_m}}{N_o} \right)$$

and therefore

$$\hat{m}(r) = \arg \max_m \frac{1}{\pi N_o} \exp \left\{ -\frac{1}{N_o}(|r|^2 + \mathcal{E}_m) \right\} I_0 \left( \frac{2|r|\sqrt{\mathcal{E}_m}}{N_o} \right)$$

In this case $I_0(x)$ is increasing with $x$ while the exponential decreases with $x$, and so there are no further simplifications. Nevertheless, we still see that the estimate does not depend on the phase of the signal, and therefore only amplitude information is useful in distinguishing the signals.

8. [BPSK versus QPSK with Phase Error]

We saw in the previous problem that noncoherent detection of linearly modulated signals is not a good idea. In this problem we consider the situation where the phase is estimated at the receiver (say, using a phase-locked loop) but there is a residual phase error $\phi$, which is not known at the receiver. As before, the sufficient statistic for decision making is given by:

$$R = \sqrt{\mathcal{E}_m} e^{j\phi} + W$$

The receiver does not know that there is a phase error, and so it makes decisions assuming that $\phi$ is equal to zero. Clearly the presence of the phase error should affect the error probability at the output of the detector.

(a) Show that for BPSK

$$P_b = Q\left(\sqrt{2\gamma_b \cos^2 \phi}\right)$$

Solution: The decision regions are the same as standard BPSK since the receiver thinks the phase is zero. However, the actual constellation points are at $\sqrt{\mathcal{E}} e^{j\phi}$ and $-\sqrt{\mathcal{E}} e^{j\phi}$. It is not hard to see that the error probability is the same no matter which symbol is sent. Thus

$$P_b = P_e = P\{R_I \leq 0 | \{1 \text{ sent}\}\} = P\{\sqrt{\mathcal{E}} \cos \phi + W_I \leq 0\} = Q\left(\sqrt{2\gamma_b \cos^2 \phi}\right)$$

(b) Show that for QPSK with Gray coding

$$P_b = \frac{1}{2} Q\left(\sqrt{4\gamma_b \sin^2 (\phi + \pi/4)}\right) + \frac{1}{2} Q\left(\sqrt{4\gamma_b \cos^2 (\phi + \pi/4)}\right)$$

Solution: Here again the decision regions are the same as standard QPSK, and the constellation points are at $\sqrt{\mathcal{E}} e^{j\pi/4} e^{j\phi}$, $\sqrt{\mathcal{E}} e^{j3\pi/4} e^{j\phi}$, $\sqrt{\mathcal{E}} e^{j5\pi/4} e^{j\phi}$, and $\sqrt{\mathcal{E}} e^{j7\pi/4} e^{j\phi}$, corresponding to 00, 01, 11, 10 (Gray coding). Again it is not difficult to see that the probability of error for the left and right bits are different no matter which bit pair is transmitted and one of them is given by (since $\mathcal{E} = 2\mathcal{E}_b$ in this case)

$$P_{b,1} = P\{R_Q \leq 0 | \{00 \text{ sent}\}\} = P\{\sqrt{\mathcal{E}} \sin (\phi + \pi/4) + W_Q \leq 0\} = Q\left(\sqrt{4\gamma_b \sin^2 (\phi + \pi/4)}\right)$$
and the other is given by

\[
P_{b,2} = P(R_I \leq 0|R_{00} = 0) = P(\sqrt{E} \cos(\phi + \pi/4) + W_Q \leq 0) = Q\left(\sqrt{4\gamma_b \cos^2(\phi + \pi/4)}\right)
\]

and the result follows.

(c) Using Matlab, compare \(P_b\) for BPSK and QPSK for \(\gamma_b = 10\) and \(\phi = 0.1, 0.2, 0.3\) radians. Which modulation scheme is more sensitive to phase errors?

**Solution:** Table shows bit error probabilities for both methods (note that BPSK and QPSK have the same performance when \(\phi = 0\)):

<table>
<thead>
<tr>
<th>(\phi)</th>
<th>BPSK</th>
<th>QPSK</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>4.298 \times 10^{-6}</td>
<td>1.59 \times 10^{-5}</td>
</tr>
<tr>
<td>0.2</td>
<td>5.853 \times 10^{-6}</td>
<td>1.188 \times 10^{-4}</td>
</tr>
<tr>
<td>0.3</td>
<td>9.669 \times 10^{-6}</td>
<td>7.924 \times 10^{-4}</td>
</tr>
</tbody>
</table>

QPSK is much more sensitive to phase offset.