ECE 562: Problem Set 3

Probability of Error for Linear Modulation, Orthogonal Modulation

Due: Tuesday September 22 in class

Reading: Lecture Notes 8-10; Sections 4.1 and 5.4 of Proakis & Salehi; Chapter 4 of Madhow

Reminder: Exam 1 will be held on Thursday, October 8 from 7-8:20 PM in 2013 ECEB. It will cover the material for HW1-HW3. You will be allowed one sheet of notes (8.5” × 11”, both sides) for the exam. Otherwise the exam is closed book.

1. [Performance of MPSK]

(a) Using the Intelligent Union Bound, show that the symbol error probability for MPSK signaling in AWGN is bounded by

\[ P_e \leq 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin \frac{\pi}{M}\right). \]

(b) Now, derive the following exact expression for \( P_e \).

\[ P_e = \frac{1}{\pi} \int_0^{(M-1)\pi/M} \exp\left[-\frac{E_s}{N_0} \frac{\sin^2(\pi/M)}{\sin^2 \theta}\right] d\theta. \]

Hint: One way to proceed is to shift the origin to the signal point under consideration and use polar co-ordinates with the appropriate limits of integration.

2. [Gray coding for QPSK]

Consider the following two bit assignments for QPSK

\[
\begin{array}{ccc}
01 & \bullet & 00 \\
11 & \bullet & 10 \\
\end{array} \quad \begin{array}{ccc}
01 & \bullet & 00 \\
10 & \bullet & 11 \\
\end{array}
\]

(1) (2)

(a) Show that assignment (1), which corresponds to Gray coding, results in an average bit error probability of \( P_b = Q(\sqrt{2\gamma_b}) \).

(b) Show that under assignment (2), the first bit (from the left) sees an average probability of error of \( Q(\sqrt{2\gamma_b}) \), whereas the second bit sees an average probability of error of \( 2Q(\sqrt{2\gamma_b})\left[1 - Q(\sqrt{2\gamma_b})\right] \).

Thus

\[ P_b = \frac{1}{2} Q(\sqrt{2\gamma_b}) + Q(\sqrt{2\gamma_b})[1 - Q(\sqrt{2\gamma_b})] \geq Q(\sqrt{2\gamma_b}) \]

3. [NNA versus IUB]

Consider the 8-ary constellation shown below:
4. [Gray Coding and Bit Error Probability]
Consider the 8-ary QAM constellation shown below (where all nearest neighbors are equidistant):

(a) Determine whether you can label the signal points using three bits so that nearest neighbors differ by at most one bit (Gray coding). If so, find such a labeling. If not, state why not and find a labeling that minimizes the maximum number of bit transitions between neighbors.

(b) For the labelings found in part (a), compute the nearest neighbor approximation for the average bit error probability $P_b$ as a function of $d^2/N_0$.

5. [“Semi-Orthogonal” Signal Set]
Consider the signal set with $M = 2N$ signals given by:

$$s_m(t) = \begin{cases} 
\sqrt{E_g} g_m(t) & m = 0, 1, \ldots, N - 1 \\
 j \sqrt{E_g} g_{m-N}(t) & m = N, \ldots, M - 1.
\end{cases}$$

where $\{g_k(t)\}_{k=1}^N$ are real-valued orthonormal functions. Clearly this signal set satisfies: $\Re[\rho_{k,\ell}] = 0$, for $k \neq \ell$, but not $\rho_{k,\ell} = 0$, for $k \neq \ell$.

(a) Argue that $R_k = \langle r(t), g_k(t) \rangle$, $k = 0, 1, \ldots, N - 1$, form sufficient statistics for optimal decision making at the receiver for an AWGN channel.

(b) Now define the $M$ real-valued statistics

$$y_m = \begin{cases} 
\frac{r_{m,I}}{2} & m = 0, 1, \ldots, N - 1 \\
\frac{r_{(m-N),Q}}{2} & m = N, \ldots, M - 1.
\end{cases}$$
Show that the MPE decision rule is given by
\[ \hat{m}_{\text{MPE}} = \arg \max_m y_m \]

(c) Find an expression for \( P_e \) for the MPE decision rule.

6. [Asymptotic Performance of Orthogonal Signaling]
In this problem you will show the following result for \( M \)-ary orthogonal modulation
\[ \lim_{M \to \infty} P_c = \begin{cases} 1 & \gamma_b > \ln 2 \\ 0 & \gamma_b < \ln 2 \end{cases} \]
where \( P_c \) is the probability of correct decision making.
Recall that we showed in class that
\[ P_c = \int_{-\infty}^{\infty} \left[ 1 - Q(x + \sqrt{2\gamma_b \log_2 M}) \right]^{M-1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \]
(a) Show that for any \( x \),
\[ \lim_{M \to \infty} \left[ 1 - Q(x + \sqrt{2\gamma_b \log_2 M}) \right]^{M-1} = \begin{cases} 1 & \gamma_b > \ln 2 \\ 0 & \gamma_b < \ln 2 \end{cases} \]
Hint: Use L’Hôpital’s rule on the log of the expression before taking the limit.
(b) Use the result of part (a) to arrive at the desired result.

7. [Noncoherent Demodulation of Linearly Modulated Signals]
The received signal for one symbol period for linear memoryless modulation on an ideal AWGN channel is given by:
\[ r(t) = \sqrt{E_m} e^{j\theta_m} g(t) e^{j\phi} + w(t) \]
where the phase offset \( \phi \) is due to the delay introduced by channel. If \( \phi \) is known at the receiver, we can correct for it (by projecting \( y(t) \) on \( g(t) e^{j\phi} \) to produce the sufficient statistic) and suffer no loss in detection performance. However, if \( \phi \) is not known, we may project \( r(t) \) on \( g(t) \) to get the sufficient statistic
\[ R = \sqrt{E_m} e^{j\theta_m} e^{j\phi} + W \]
where \( W \sim \mathcal{CN}(0, N_0) \). Since \( \phi \) is not of direct interest to the receiver, we treat it as a nuisance parameter. As we saw in class, there are two ways to deal with such parameters.
(a) Assume that \( \phi \in [0, 2\pi] \), and find \( \hat{m}_{\text{JML}}(r) \) using the joint ML approach. Interpret your answer.
(b) Now assume that \( \phi \) is a random variable that is uniformly distributed on \( [0, 2\pi] \), and find \( \hat{m}_{\text{MAP}}(y) \). Simplify your answer as much as possible (note that your answer can be written in terms of the Bessel function \( I_0 \)).
Note: You should see from this problem that noncoherent demodulation of linearly modulated signals is not a very good idea.

8. [BPSK versus QPSK with Phase Error]
We saw in the previous problem that noncoherent detection of linearly modulated signals is not a good idea. In this problem we consider the situation where the phase is estimated at the receiver (say, using a phase-locked loop) but there is a residual phase error \( \phi \), which is not known at the receiver. As before, the sufficient statistic for decision making is given by:
\[ R = \sqrt{E_m} e^{j\theta_m} e^{j\phi} + W \]
The receiver does not know that there is a phase error, and so it makes decisions assuming that \( \phi \) is equal to zero. Clearly the presence of the phase error should affect the error probability at the output of the detector.
(a) Show that for BPSK

\[ P_b = Q\left(\sqrt{2\gamma_b \cos^2 \phi}\right) \]

(b) Show that for QPSK with Gray coding

\[ P_b = \frac{1}{2} Q\left(\sqrt{4\gamma_b \sin^2 (\phi + \pi/4)}\right) + \frac{1}{2} Q\left(\sqrt{4\gamma_b \cos^2 (\phi + \pi/4)}\right) \]

(c) Using Matlab, compare \( P_b \) for BPSK and QPSK for \( \gamma_b = 10 \) and \( \phi = 0.1, 0.2, 0.3 \) radians. Which modulation scheme is more sensitive to phase errors?