ECE 562: Problem Set 2: Solutions
Digital Modulation, Complex WGN, Optimum Receiver in WGN

1. **[Simplex Signal Set]**

For any $m$, 

$$
\|s'_m\|^2 = \left\langle s_m - \frac{1}{M} \sum_{i=1}^{M} s_i, s_m - \frac{1}{M} \sum_{j=1}^{M} s_j \right\rangle
$$

$$
= \langle s_m, s_m \rangle - \left\langle \frac{1}{M} \sum_{i=1}^{M} s_i, s_m \right\rangle - \left\langle s_m, \frac{1}{M} \sum_{j=1}^{M} s_j \right\rangle + \left\langle \frac{1}{M} \sum_{i=1}^{M} s_i, \frac{1}{M} \sum_{j=1}^{M} s_j \right\rangle
$$

$$
= \|s_m\|^2 - \frac{1}{M} \sum_{i=1}^{M} \langle s_i, s_m \rangle - \frac{1}{M} \sum_{j=1}^{M} \langle s_m, s_j \rangle + \frac{1}{M^2} \sum_{i=1}^{M} \sum_{j=1}^{M} \langle s_i, s_j \rangle
$$

$$
= \|s_m\|^2 - \frac{2}{M} \|s_m\|^2 + \frac{1}{M^2} \sum_{i=1}^{M} \|s_i\|^2 = \left( 1 - \frac{2}{M} + \frac{1}{M} \right) \mathcal{E} = \left( 1 - \frac{1}{M} \right) \mathcal{E}
$$

For any pair $k$ and $m$ with $m \neq p$, 

$$
\langle s'_k, s'_m \rangle = \left\langle s_k - \frac{1}{M} \sum_{i=1}^{M} s_i, s_m - \frac{1}{M} \sum_{j=1}^{M} s_j \right\rangle
$$

$$
= \langle s_k, s_p \rangle - \left\langle \frac{1}{M} \sum_{i=1}^{M} s_i, s_m \right\rangle - \left\langle s_k, \frac{1}{M} \sum_{j=1}^{M} s_j \right\rangle + \left\langle \frac{1}{M} \sum_{i=1}^{M} s_i, \frac{1}{M} \sum_{j=1}^{M} s_j \right\rangle
$$

$$
= -\frac{1}{M} \sum_{i=1}^{M} \langle s_i, s_m \rangle - \frac{1}{M} \sum_{j=1}^{M} \langle s_k, s_j \rangle + \frac{1}{M} \sum_{i=1}^{M} \sum_{j=1}^{M} \langle s_i, s_j \rangle
$$

$$
= -\frac{1}{M} \|s_m\|^2 - \frac{1}{M} \|s_k\|^2 + \frac{1}{M^2} \sum_{i=1}^{M} \|s_i\|^2 = \left( -\frac{2}{M} + \frac{1}{M} \right) \mathcal{E} = -\frac{\mathcal{E}}{M}
$$

Thus 

$$
\rho_{km} = \frac{\langle s'_k, s'_m \rangle}{\|s'_k\| \|s'_m\|} = \frac{-\frac{\mathcal{E}}{M}}{(1 - \frac{1}{M}) \mathcal{E}} = -\frac{1}{M - 1}
$$

2. **[Signal Constellation Optimization]**

(a) To find $d_{\text{min}}^2$ we need to consider two cases:

i. $d_{\text{min}}^2$ is achieved by two points on the inner circle

ii. $d_{\text{min}}^2$ is achieved by a point on the inner circle and a point on the outer circle

In the first case, $2d_{\text{min}}^2 = 2^2$ by examining the inscribed square, so $d_{\text{min}}^2 = 2$. In the second case, 

$$
d_{\text{min}}^2 = \|(0, a) - (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})\|^2 = a^2 - a\sqrt{2} + 1
$$

The crossover between these two expression occurs when they are equal, which occurs when 

$$
a^2 - a\sqrt{2} + 1 = 2 \Rightarrow a^* = \frac{\sqrt{2} + \sqrt{6}}{2}
$$
Therefore

\[ d_{min}^2(a) = \begin{cases} 
2 & \text{if } a > a^*, \\
\frac{a^2 - a\sqrt{2} + 1}{2} & \text{if } a \leq a^* 
\end{cases} \]

Furthermore, for this constellation

\[ E_s = \frac{1}{8} \left( 4(1^2) + 4(a^2) \right) = \frac{1 + a^2}{2} \]

\[ E_b = \frac{E_s}{\log_2 8} = \frac{1 + a^2}{6} \]

Therefore

\[ \zeta(a) = \frac{d_{min}^2(a)}{E_b} = \begin{cases} 
\frac{12}{2 + a^2} & \text{if } a > a^*, \\
\frac{6(a^2 - a\sqrt{2} + 1)}{4 + a^2} & \text{if } a \leq a^* 
\end{cases} \]

(b) The measure of goodness \( \zeta(a) \) is monotone decreasing for \( a > a^* \) and monotone increasing for \( 1 \leq a \leq a^* \). Therefore \( a^* = \frac{\sqrt{2} + \sqrt{6}}{2} \).

(c) For the optimum value of \( a \), we have

\[ \zeta(a^*) = 2(3 - \sqrt{3}) \approx 2.536 \]

For 8-ary PAM,

\[ E_m = \left( 2m - 1 - 8 \right)^2 \frac{d^2}{4} \]

\[ E_s = \frac{1}{8} \sum_{m=1}^{8} \left( 2m - 1 - 8 \right)^2 \frac{d^2}{4} = \frac{21}{4} d^2 \]

\[ E_b = \frac{E_s}{\log_2 8} = \frac{4}{7} d^2 \]

\[ \zeta_{8-PAM} = \frac{d^2}{E_b} = \frac{4}{7} = 0.5714 \]

The proposed constellation is considerably better than 8-PAM in terms of \( \zeta \).

3. **[Competing Signal Constellations]**

   For 8-PSK from the notes we have,

\[ \zeta_{8-PSK} = 2 \log_2 8 \left( 1 - \cos \frac{2\pi}{8} \right) = 3(2 - \sqrt{2}) \approx 1.76 \]

For rectangular QAM (either horizontal or vertical), it is easy to see that:

\[ E_s = \frac{1}{8} \left[ 4\mathcal{E} + 4.5\mathcal{E} \right] = 3\mathcal{E} \implies \mathcal{E}_b = \mathcal{E}, \quad \text{and} \quad d_{min}^2 = 2\mathcal{E}. \]

Thus \( \zeta_{8-QAM} = 2 \), and 8-QAM is slightly better than 8-PSK in terms of \( \zeta \).

4. **[Complex Random Vector]**
\( \Sigma = \mathbb{E} \left[ (Y - m_Y) (Y - m_Y)^\dagger \right] \)
\[
= \mathbb{E} \left[ \left( Y_t + jY_Q - (m_Y + jm_Y_Q) \right) \left( Y_t + jY_Q - (m_Y + jm_Y_Q) \right)^\dagger \right]
\]
\[
= \mathbb{E} \left[ \left( Y_t - m_Y + j \left( Y_Q - m_Y_Q \right) \right) \left( Y_t - m_Y + j \left( Y_Q - m_Y_Q \right) \right)^\dagger \right]
\]
\[
= \mathbb{E} \left[ \left( Y_t - m_Y + j \left( Y_Q - m_Y_Q \right) \right) \left( (Y_t - m_Y)^\dagger - j \left( Y_Q - m_Y_Q \right)^\dagger \right) \right]
\]
\[
= \mathbb{E} \left[ (Y_t - m_Y) (Y_t - m_Y)^\dagger + \left( Y_Q - m_Y_Q \right) \left( Y_Q - m_Y_Q \right)^\dagger \right]
\]
\[
+ j \mathbb{E} \left[ \left( \left( Y_Q - m_Y_Q \right) (Y_t - m_Y) \right)^\dagger - \left( Y_Q - m_Y_Q \right) (Y_Q - m_Y_Q)^\dagger \right] \]
\[
= (\Sigma_I + \Sigma_Q) + j (\Sigma_{IQ} - \Sigma_{QI}) \]

\( \bar{\Sigma} = \mathbb{E} \left[ (Y - \bar{m}_Y) (Y - \bar{m}_Y)^\dagger \right] \)
\[
= \mathbb{E} \left[ \left( Y_t + jY_Q - (m_Y + jm_Y_Q) \right) \left( Y_t + jY_Q - (m_Y + jm_Y_Q) \right)^\dagger \right]
\]
\[
= \mathbb{E} \left[ \left( Y_t - m_Y + j \left( Y_Q - m_Y_Q \right) \right) \left( Y_t - m_Y + j \left( Y_Q - m_Y_Q \right) \right)^\dagger \right]
\]
\[
= \mathbb{E} \left[ \left( Y_t - m_Y + j \left( Y_Q - m_Y_Q \right) \right) \left( (Y_t - m_Y)^\dagger + j \left( Y_Q - m_Y_Q \right)^\dagger \right) \right]
\]
\[
= \mathbb{E} \left[ (Y_t - m_Y) (Y_t - m_Y)^\dagger - \left( Y_Q - m_Y_Q \right) \left( Y_Q - m_Y_Q \right)^\dagger \right]
\]
\[
+ j \mathbb{E} \left[ \left( \left( Y_Q - m_Y_Q \right) (Y_t - m_Y) \right)^\dagger + \left( Y_t - m_Y \right) \left( Y_Q - m_Y_Q \right)^\dagger \right] \]
\[
= (\Sigma_I - \Sigma_Q) + j (\Sigma_{QI} - \Sigma_{IQ}) \]

5. [WGN in Complex Baseband]

(a) We can express \( Z \) as:

\[
Z = \int_0^1 r(t) \sin \pi t \, dt - \int_0^1 (s(t) + w(t)) \sin \pi t \, dt \]
\[
= \int_0^1 (\sin^2 \pi t + j \sin \pi t \cos \pi t) \, dt + \int_0^1 w(t) \sin \pi t \, dt = \frac{1}{2} + \langle w, \psi \rangle \]

where \( \psi(t) = \sin \pi t \). Since \( \|\psi\|^2 = \frac{1}{2} \), \( \langle w, \psi \rangle \sim \mathcal{CN}(0, \frac{N_o}{2}) \), with \( N_o = 2 \). Therefore, \( Z \sim \mathcal{CN}(\frac{1}{2}, 1) \), \( Z_I \sim \mathcal{N}(\frac{1}{2}, \frac{1}{2}) \), and \( Z_Q \sim \mathcal{N}(0, \frac{1}{2}) \) with \( Z_I \) and \( Z_Q \) independent.

Therefore

\[
P[Z_I \geq 2] = Q \left( \frac{2 - \frac{1}{2}}{\sqrt{1/2}} \right) = Q \left( \frac{3}{\sqrt{2}} \right) \]

(b) Since \( Z_I \) and \( Z_Q \) are independent, \( Z_I + 2Z_Q \sim \mathcal{N}(\frac{1}{2}, \frac{1}{2} + 2) \). Therefore

\[
P[Z_I + 2Z_Q \geq 1] = Q \left( \frac{1 - \frac{1}{2}}{\sqrt{5/2}} \right) = Q \left( \frac{1}{\sqrt{10}} \right) \]
(c) Again since $Z_I$ and $Z_Q$ are independent

$$P\{Z_I \geq 1, Z_Q \geq 2\} = P\{Z_I \geq 1\}P\{Z_Q \geq 2\} = Q\left(\frac{1 - \frac{1}{2}}{\sqrt{1/2}}\right)Q\left(\frac{2 - 0}{\sqrt{1/2}}\right) = Q\left(\frac{1}{\sqrt{2}}\right)Q\left(2\sqrt{2}\right)$$

6. [Unequal Priors]

For BPSK we have points in the constellation at $\pm \sqrt{E}$. The conditional pdfs of $R$ are given by

$$p_0(r) = \frac{1}{\pi N_o} \exp\{-|r - \sqrt{E}|^2/N_o\}, \quad p_1(r) = \frac{1}{\pi N_o} \exp\{-|r + \sqrt{E}|^2/N_o\}$$

The MPE decision region for 0 is given by

$$\Gamma_0 = \{ r : \pi_0 p_0(r) > \pi_1 p_1(r) \} = \left\{ r : \frac{\pi_0 p_0(r)}{\pi_1 p_1(r)} > 1 \right\} = \left\{ r : \frac{\pi_0}{\pi_1} \exp\left\{\frac{4\sqrt{E}r_I}{N_o}\right\} > 1 \right\} = \left\{ r : r_I > -\frac{N_o}{4\sqrt{E}} \ln\left(\frac{\pi_0}{\pi_1}\right) \right\}$$

and $\Gamma_1 = \Gamma_0$.

7. [Probability of error for PAM]

Assuming that the distance between constellation points is $d$, the average signal energy is

$$E_s = \frac{1}{4} \left(\frac{d^2}{4} + \frac{9d^2}{4} + \frac{25d^2}{4} + \frac{49d^2}{4}\right) = \frac{21d^2}{4}$$

and therefore $d = \sqrt{\frac{4E_s}{21}}$.

We need to consider the case of exterior and interior points separately. Assuming equal priors, we apply ML detection. The decision regions are vertical strips in the I-Q plane, with the six interior points all having the same probability of error, and the two exterior points having the same probability of error.

It is easier to compute probabilities of correct decisions first. For an interior point

$$P_{c}^{\text{interior}} = P\left\{ -\frac{d}{2} \leq W_I \leq \frac{d}{2} \right\} = 1 - 2P\left\{ W_I > \frac{d}{2} \right\} = 1 - 2Q\left(\frac{d/2}{\sqrt{N_o/2}}\right) = 1 - 2Q\left(\sqrt{\frac{d^2}{2N_o}}\right)$$

For an exterior point

$$P_{c}^{\text{exterior}} = P\left\{ W_I \geq -\frac{d}{2} \right\} = Q\left(\frac{-d/2}{\sqrt{N_o/2}}\right) = 1 - Q\left(\frac{d/2}{\sqrt{N_o/2}}\right) = 1 - Q\left(\sqrt{\frac{d^2}{2N_o}}\right)$$

Then the probability of a correct decision is

$$P_c = \frac{1}{8} (6P_{c}^{\text{interior}} + 2P_{c}^{\text{exterior}}) = 1 - \frac{7}{4} Q\left(\sqrt{\frac{d^2}{2N_o}}\right)$$

Therefore the probability of error is

$$P_e = \frac{7}{4} Q\left(\sqrt{\frac{d^2}{2N_o}}\right) = \frac{7}{4} Q\left(\sqrt{\frac{2E_s}{21N_o}}\right)$$