1. **[Random process with exponential sample paths]**

(a) Since $X$ takes on values in $[0,1]$, $Y(t_0)$ takes values only in $[e^{-t_0},1]$. Then for $e^{-t_0} \leq y \leq 1$:

$$F_{Y(t_0)}(y) = P\{Y(t_0) \leq y\} = P\{e^{-t_0}X \leq y\} = P\{X \geq -(\ln y)/t_0\} = 1 + (\ln y)/t_0$$

The cdf is given by:

$$F_{Y(t_0)}(t_0)(y) = \begin{cases} 
0 & \text{if } y \leq e^{-t_0}, \\
1 + (\ln y)/t_0 & \text{if } e^{-t_0} \leq y \leq 1, \\
1 & \text{if } y \geq 1 
\end{cases}$$

Differentiating we get

$$f_{Y(t_0)}(y) = \begin{cases} 
\frac{1}{t_0y} & \text{if } e^{-t_0} \leq y \leq 1, \\
0 & \text{else} 
\end{cases}$$

(b) $m_Y(t) = E[Y(t)] = E[e^{-tX}] = \int_0^1 e^{-tx} dx = \frac{1 - e^{-t}}{t}$

$$R_Y(s,t) = E[Y(s)Y(t)] = E[e^{-(t+s)X}] = \frac{1 - e^{-(t+s)}}{t+s}$$

(c) No. The mean changes with $t$.

2. **[Signaling in AWGN]**

(a) $Y_1 = \int_0^T \sin^2(\pi t/T) dt + \int_0^T n(t) \sin(\pi t/T) dt = \frac{T}{2} + Z_1$

where $Z_1 = \int_0^T n(t) \sin(\pi t/T) dt$ is zero-mean Gaussian with

$$\text{Var}(Z_1) = \text{Cov}(Z_1, Z_1) = \frac{N_0}{2} \int_0^T \sin^2(\pi t/T) dt = \frac{N_0 T}{4}$$

Therefore

$$P\{Y_1 > \gamma\} = P\left\{ Z_1 > \gamma - \frac{T}{2} \right\} = Q\left( \frac{\gamma - \frac{T}{2}}{\sqrt{\frac{N_0 T}{4}}} \right)$$

(b) $Y_2 = \int_0^T \sin(\pi t/T) \cos(\pi t/T) dt + \int_0^T n(t) \cos(\pi t/T) dt = \frac{1}{2} \int_0^T \sin(2\pi t/T) dt + Z_2 = Z_2$

where $Z_2 = \int_0^T n(t) \cos(\pi t/T) dt$ is zero-mean Gaussian with

$$\text{Var}(Z_2) = \text{Cov}(Z_2, Z_2) = \frac{N_0}{2} \int_0^T \cos^2(\pi t/T) dt = \frac{N_0 T}{4}$$

Therefore

$$P\{Y_2 > \gamma\} = P\{Z_2 > \gamma\} = Q\left( \frac{\gamma}{\sqrt{\frac{N_0 T}{4}}} \right)$$
(c) The random variable $X = Y_1 + Y_2$ is Gaussian with $E[X] = \frac{T}{2}$ and

$$\text{Var}(X) = \text{Cov}(Y_1 + Y_2, Y_1 + Y_2) = \text{Cov}(Z_1 + Z_2, Z_1 + Z_2) = \text{Var}(Z_1) + \text{Var}(Z_2) + 2\text{Cov}(Z_1, Z_2)$$

Now

$$\text{Cov}(Z_1, Z_2) = \frac{N_0}{2} \int_0^T \sin(\pi t/T) \cos(\pi t/T) dt = 0$$

Therefore

$$\text{Var}(X) = \text{Var}(Z_1) + \text{Var}(Z_2) = \frac{N_0 T}{2}$$

and

$$P\{Y_1 + Y_2 > \gamma\} = P\{X > \gamma\} = \Phi\left(\frac{\gamma - \frac{T}{2}}{\sqrt{\frac{N_0 T}{2}}}\right)$$

3. [Passband to baseband]

(a) We showed in class (also see notes) that

$$\tilde{s}(t) = \sqrt{2} s_I(t) \cos(2\pi f_c t) - \sqrt{2} s_Q(t) \sin(2\pi f_c t)$$

For the in-phase portion:

$$\tilde{s}(t)(\sqrt{2} \cos(2\pi f_c t)) = 2s_I(t) \cos^2(2\pi f_c t) - 2s_Q(t) \sin(2\pi f_c t) \cos(2\pi f_c t)$$

Applying the trig identities $2\cos^2(x) = 1 + \cos(2x)$ and $2\sin(x) \cos(x) = \sin(2x)$:

$$\tilde{s}(t)(\sqrt{2} \cos(2\pi f_c t)) = s_I(t) + s_I(t) \cos(4\pi f_c t) - s_Q(t) \sin(4\pi f_c t)$$

Lowpass filtering gets rid of the double frequency terms, and so:

$$\left[\tilde{s}(t)(\sqrt{2} \cos(2\pi f_c t))\right]_{\text{LPF}} = s_I(t)$$

For the quadrature portion:

$$\tilde{s}(t)(-\sqrt{2} \sin(2\pi f_c t)) = -2s_I(t) \sin(2\pi f_c t) \cos(2\pi f_c t) + 2s_Q(t) \sin^2(2\pi f_c t)$$

Applying the trig identities $2\sin^2(x) = 1 - \cos(2x)$ and $2\sin(x) \cos(x) = \sin(2x)$:

$$\tilde{s}(t)(-\sqrt{2} \sin(2\pi f_c t)) = -s_I(t) \sin(4\pi f_c t) + s_Q(t) \cos(4\pi f_c t)$$

Lowpass filtering gets rid of the double frequency terms:

$$\left[\tilde{s}(t)(\sqrt{2} \sin(2\pi f_c t))\right]_{\text{LPF}} = s_Q(t)$$

(b) We can expand the expression on the right as

$$\tilde{s}(t) = g(t)(\cos(2\pi f_c t) \cos(\frac{\pi}{4}) - \sin(2\pi f_c t) \sin(\frac{\pi}{4})) + f(t)(\sin(2\pi f_c t) \cos(\frac{\pi}{4}) + \cos(2\pi f_c t) \sin(\frac{\pi}{4}))$$

$$= \frac{g(t) + f(t)}{\sqrt{2}} \cos(2\pi f_c t) - \frac{g(t) - f(t)}{\sqrt{2}} \sin(2\pi f_c t)$$

from which we can conclude that

$$s_I(t) = \frac{g(t) + f(t)}{2} \quad s_Q(t) = \frac{g(t) - f(t)}{2}$$
4. [Baseband representation of channel]

\[ H(f) = \frac{Y(f)}{X(f)} = \frac{\hat{Y}_+(f + f_c)}{\hat{X}_+(f + f_c)} = \frac{\sqrt{2}\hat{Y}(f + f_c)\mathbb{1}_{\{f + f_c \geq 0\}}}{\sqrt{2}\hat{X}(f + f_c)\mathbb{1}_{\{f + f_c \geq 0\}}} = \hat{H}(f + f_c)\mathbb{1}_{\{f + f_c \geq 0\}} = \frac{1}{\sqrt{2}}\hat{H}_+(f + f_c) \]

5. [Complex Gaussian Preview]

First note that

\[ Y_I + jY_Q = (X_I + jX_Q)(\cos \phi + j \sin \phi) = (X_I \cos \phi - X_Q \sin \phi) + j(X_I \sin \phi + X_Q \cos \phi) \]

Therefore

\[ \begin{bmatrix} Y_I \\ Y_Q \end{bmatrix} = \begin{bmatrix} \cos \phi & -\sin \phi \\ \sin \phi & \cos \phi \end{bmatrix} \begin{bmatrix} X_I \\ X_Q \end{bmatrix} \]

Let the matrix on the right hand side be denoted by \( A \). Since \((X_I, X_Q)\) are jointly Gaussian with zero mean and covariance matrix \( \Sigma_X = I \), we have that \((Y_I, Y_Q)\) are also jointly Gaussian with zero mean and covariance matrix given by

\[ \Sigma_Y = A \Sigma_X A^T = AA^T = I \]

6. [Signal Space]

(a) Applying the definition of the inner product

\[ \langle f, g \rangle = \int_0^4 f(t)g^*(t)dt \]

gives

\[ \langle f_1, f_2 \rangle = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) (2) + \left( \frac{1}{2} \right) \left( -\frac{1}{2} \right) (2) = 0 \]

\[ \langle f_1, f_3 \rangle = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) (1) + \left( \frac{1}{2} \right) \left( -\frac{1}{2} \right) (2) + \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) (1) = 0 \]

\[ \langle f_2, f_3 \rangle = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) (1) + \left( \frac{1}{2} \right) \left( -\frac{1}{2} \right) (1) + \left( -\frac{1}{2} \right) \left( -\frac{1}{2} \right) (1) + \left( -\frac{1}{2} \right) \left( \frac{1}{2} \right) (1) = 0 \]

It is easy to see that the \( f_i \)'s are all unit norm and therefore they are orthonormal.

(b) The \( f_i \)'s form an orthonormal basis for the space \( \mathcal{S} = \text{span}\{f_1, f_2, f_3\} \), but we do not know \textit{a priori} if \( s(t) \) belongs to \( \mathcal{S} \). We can project \( s(t) \) onto \( \mathcal{S} \) and if the projection equals \( s(t) \), then we will have expressed \( s(t) \) as a linear combination of the \( f_i \)'s. To project we need to compute \( \langle s, f_i \rangle \) for \( i = 1, 2, 3 \):

\[ \langle s, f_1 \rangle = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) (1) + \left( \frac{1}{2} \right) \left( -\frac{1}{4} \right) (1) + \left( \frac{1}{2} \right) \left( \frac{1}{4} \right) (1) = \frac{1}{4} \]

\[ \langle s, f_2 \rangle = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) (1) + \left( -\frac{1}{2} \right) \left( -\frac{1}{4} \right) (1) + \left( -\frac{1}{2} \right) \left( \frac{1}{4} \right) (1) = \frac{1}{4} \]

\[ \langle s, f_3 \rangle = \left( \frac{1}{2} \right) \left( \frac{1}{2} \right) (1) + \left( -\frac{1}{2} \right) \left( -\frac{1}{4} \right) (1) + \left( \frac{1}{2} \right) \left( \frac{1}{4} \right) (1) = \frac{1}{2} \]

Plotting will show that

\[ s(t) = \sum_{i=1}^3 \langle s, f_i \rangle f_i(t) = \frac{1}{4} f_1(t) + \frac{1}{4} f_2(t) + \frac{1}{2} f_3(t) \]
7. **[Gram-Schmidt]** First, note that \( s_1(t) = s_2(t) - s_3(t) \), and therefore \( s_1 \in \text{span}\{s_2, s_3\} \), which means that \( \dim(S) \leq 3 \). To begin set

\[
g_1(t) = s_2(t)
\]

Then \( \|g_1\| = \sqrt{2^2} = 2 \), and therefore

\[
f_1(t) = \frac{g_1(t)}{\|g_1\|} = \mathbb{I}_{\{0 \leq t \leq 1\}}
\]

Continuing, we set

\[
g_2(t) = s_3(t) - \langle s_3, f_1 \rangle f_1(t)
\]

However, \( s_3 \) and \( f_1 \) are never both non-zero for the same value of \( t \), so \( \langle s_3, f_1 \rangle = 0 \) and

\[
g_2(t) = s_3(t) = -2\mathbb{I}_{\{1 \leq t \leq 3\}}
\]

Then \( \|g_2\| = \sqrt{(-2)^2} = 2\sqrt{2} \), and therefore

\[
f_2(t) = \frac{g_2(t)}{\|g_2\|} = -\frac{1}{\sqrt{2}}\mathbb{I}_{\{1 \leq t \leq 3\}}
\]

Finally,

\[
g_3(t) = s_4(t) - \langle s_4, f_1 \rangle f_1(t) - \langle s_4, f_2 \rangle f_2(t)
\]

The inner products are

\[
\langle s_4, f_1 \rangle = (2) (1) = 2
\]

\[
\langle s_4, f_2 \rangle = \left( -\frac{1}{\sqrt{2}} \right) (2) = -\sqrt{2}
\]

and so we get

\[
g_3(t) = s_4(t) - 2f_1(t) + \sqrt{2}f_2(t) = \mathbb{I}_{\{1 \leq t \leq 2\}} - \mathbb{I}_{\{2 \leq t \leq 3\}}
\]

Then \( \|g_3\| = \sqrt{(1)^2(1) + (-1)^2(1)} = \sqrt{2} \), and therefore

\[
f_3(t) = \frac{g_3(t)}{\|g_3\|} = \frac{1}{\sqrt{2}}\mathbb{I}_{\{1 \leq t \leq 2\}} - \frac{1}{\sqrt{2}}\mathbb{I}_{\{2 \leq t \leq 3\}}
\]

Since \( f_1, f_2, f_3 \) are all nonzero, and we know that \( \dim(S) \leq 3 \), \( f_1, f_2, f_3 \) form an orthonormal basis for \( S \). This answer is of course not unique.

8. **[Signal Spaces as Vector Spaces]**

(a) By assumption we can represent \( s \) as

\[
s(t) = \sum_{i=1}^{n} s_i f_i(t)
\]

Also \( \langle f_i, f_j \rangle = \mathbb{I}_{i=j} \), since we have an orthonormal basis, so

\[
\|s\|^2 = \int_a^b |s(t)|^2 dt = \int_a^b \left( \sum_{i=1}^{n} s_i f_i(t) \left( \sum_{j=1}^{n} s_j f_j(t) \right)^* \right) \ dt
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{n} s_i s_j^* \int_a^b f_i(t) f_j^*(t) dt = \sum_{i=1}^{n} \sum_{j=1}^{n} s_j s_i^* \langle f_i, f_j \rangle
\]

\[
= \sum_{i=1}^{n} |s_i|^2 = s^t s = \|s\|^2
\]

(1)
\[
\langle s_k, s_m \rangle = \int_a^b s_k(t)s_m^*(t)dt = \int_a^b \left( \sum_{i=1}^n s_{k,i} f_i(t) \right) \left( \sum_{j=1}^n s_{m,j} f_j(t) \right)^* dt
\]
\[
= \sum_{i=1}^n \sum_{j=1}^n s_{k,i}s_{m,j} \int_a^b f_i(t)f_j^*(t)dt = \sum_{i=1}^n \sum_{j=1}^n s_{k,i}s_{m,j} \langle f_i, f_j \rangle
\]
\[
= \sum_{i=1}^n s_{k,i}s_{m,i}
\]
\[(b)\]

(c) In a complex inner product space note that

\[
\langle u + v, a + b \rangle = \langle u, a + b \rangle + \langle v, a + b \rangle = \langle a + b, u \rangle^* + \langle a + b, v \rangle^*
\]
\[
= \langle a, u \rangle^* + \langle b, u \rangle^* + \langle a, v \rangle^* + \langle b, v \rangle^*
\]
\[
= \langle u, a \rangle + \langle u, b \rangle + \langle v, a \rangle + \langle v, b \rangle
\]

Therefore

\[
d_{km}^2 = \|s_k - s_m\|^2 = \langle s_k, s_k \rangle - \langle s_k, s_m \rangle - \langle s_m, s_k \rangle + \langle s_m, s_m \rangle
\]
\[
= \|s_k\|^2 + \|s_m\|^2 - \left( \langle s_k, s_m \rangle + \langle s_m, s_k \rangle^* \right)
\]
\[
= \|s_k\|^2 + \|s_m\|^2 - 2\|s_k\|\|s_m\| \text{Re} [\langle s_k, s_m \rangle]
\]
\[
= \|s_k\|^2 + \|s_m\|^2 - 2\|s_k\|\|s_m\| \text{Re} [\rho_{km}]
\]
\[
= \mathcal{E}_k + \mathcal{E}_m - 2\sqrt{\mathcal{E}_k\mathcal{E}_m} \text{Re} [\rho_{km}]
\]