1. [MSE of linear equalizers]
The MSE for the $k$-th symbol when a linear equalizer $c_k$ is used was defined in class to be:

$$MSE_k = E \left[ |c_k^\dagger Z - s_{m_k}|^2 \right]$$

Using the fact that

$$Z = h_k s_{m_k} + \sum_{j \neq k} h_j s_{m_j} + W$$

show that

$$MSE_k = E_s |c_k^\dagger h_k - 1|^2 + E_s \sum_{j \neq k} |c_k^\dagger h_j|^2 + N_0 \|c_k\|^2$$

2. [Minimum MSE]
Consider the MMSE equalizer $c_{k,MMSE} = \mathcal{E}_s (\mathcal{E}_s H H^\dagger + N_0 I)^{-1} h_k$. Show that MSE achieved by this equalizer (which is the minimum MSE) is given by:

$$MMSE_k = \mathcal{E}_s - \mathcal{E}_s^2 \frac{h_k^\dagger}{h_k^\dagger (\mathcal{E}_s H H^\dagger + N_0 I)^{-1} h_k}$$

3. [MMSE equalizer as SNR goes to infinity]
As we discussed in class, the MMSE equalizer becomes ill-conditioned if we set $N_0 = 0$. Interestingly, however, we can still show that the MMSE equalizer converges to the ZF equalizer in limit as $N_0 \to 0$.

(a) Show that $MSE_k$ for $c_{k,ZF}$ is given by:

$$MSE_k(c_{k,ZF}) = N_0 \|c_{k,ZF}\|^2.$$ 

(b) Using part (a) argue that, for a fixed $\mathcal{E}_s$, $MSE_k$ for $c_{k,MMSE}$ satisfies:

$$\lim_{N_0 \to 0} MSE_k(c_{k,MMSE}) = 0.$$ 

(c) Now use the results of part (b) and Problem 2 to conclude that the MMSE equalizer indeed converges to the ZF solution as $N_0 \to 0$, for fixed $\mathcal{E}_s$.

4. [MMSE equalizer maximizes SINR]
We showed in class that the SINR for the $k$-th symbol when a linear equalizer $c_k$ is used is given by:

$$SINR_k = \frac{\mathcal{E}_s |c_k^\dagger h_k|^2}{\mathcal{E}_s \sum_{j \neq k} |c_k^\dagger h_j|^2 + N_0 \|c_k\|^2}$$

Using the result of Problem 2, show that $c_{k,MMSE}$ maximizes $SINR_k$.

Hint: Consider the problem of maximizing $SINR_k$ subject to $c_k^\dagger h_k = \alpha$, and show that the achieved maximum is independent of $\alpha$. 
5. **[Maximum SINR and Minimum MSE]**
   Show that the maximum value of SINR is given by
   \[
   \text{SINR}_{k,\text{max}} = \frac{E_s}{\text{MMSE}_k} - 1
   \]

6. **[True or False]**
   Determine if the following statements are True or False. You need to provide a brief justification for
   your answer to get credit.
   
   (a) If the mobile speed is 72 km/hr, the carrier frequency is 900 MHz, and symbol rate for communication
       is 10 symbols a second, then the mobile experiences slow fading.
   
   (b) In wireless communication, if the speed of the mobile is doubled the delay spread is also doubled.
   
   (c) A wireless channel with \( \tau_{ds} = 10^{-4} \) seconds and \( f_{\text{max}} = 100 \) Hz operating with a bandwidth
       of 1.25 MHz is a flat fading channel.
   
   (d) For a fixed \( \gamma \), \( P_b \) for binary DPSK signaling over a slow flat fading channel is always larger than
       \( P_b \) without fading, irrespective of the distribution of the fading.

7. **[Useful Result]**
   Show that
   \[
   \int_0^\infty Q(\sqrt{x}) \frac{e^{-x/\gamma}}{\gamma} dx = \frac{1}{2} \left( 1 - \sqrt{\frac{\gamma}{2 + \gamma}} \right).
   \]
   **Hint:** Start with:
   \[
   \int_0^\infty Q(\sqrt{x}) \frac{e^{-x/\gamma}}{\gamma} dx = \int_0^\infty \int_{\pi}^{\infty} \frac{e^{-t^2/2}}{2\pi} \frac{e^{-x/\gamma}}{\gamma} dt \, dx = \int_0^\infty \int_{\pi}^{\infty} \frac{e^{-t^2/2}}{2\pi} \frac{e^{-x/\gamma}}{\gamma} dx \, dt
   \]
   and use the fact that the Gaussian pdf integrates to 1 to conclude the result.

You do not need to turn in the solutions to Problems 8-10. I’ll give you the solutions to these next Thursday when you turn in the solutions to Problems 1-7.

8. **[MPSK Signaling in Rayleigh Fading]**
   For MPSK signaling,
   \[
   P_e(\gamma_s) = \frac{1}{\pi} \int_0^{(M-1)\pi/M} \exp \left[ -\frac{\gamma_s \sin^2(\pi/M)}{\sin^2 \theta} \right] d\theta.
   \]
   Using this expression show that the average symbol error probability \( \overline{P}_e \) for MPSK signaling in Rayleigh
   fading is given in closed form by
   \[
   \overline{P}_e = \left( 1 - \frac{1}{M} \right) - \frac{1}{\sqrt{1 + a^2}} \left[ \frac{1}{2} + \frac{1}{\pi} \tan^{-1} \left( \frac{\cot \pi/M}{\sqrt{1 + a^2}} \right) \right],
   \]
   where \( a^2 = \frac{1}{\tau_s \sin^2 \pi/M} \).
   **Hint:** You may need to use the following integral
   \[
   \int_{\theta_1}^{\theta_2} \frac{1}{\cosec^2 \theta + a^2} d\theta = \frac{1}{a^2} \left[ \frac{1}{\sqrt{1 + a^2}} \tan^{-1} \left( \frac{\cot \theta}{\sqrt{1 + a^2}} \right) - \left( \frac{\pi}{2} - \theta \right) \right]_{\theta_1}^{\theta_2} \text{ for } 0 \leq \theta_1 \leq \theta_2 \leq \pi/2.
   \]

9. **[Diversity]**
   Consider BPSK with channel gain \( a \), i.e., the received signal is
   \[
   r(t) = \pm a \sqrt{E} g(t) + w(t), \quad 0 \leq t \leq T,
   \]
   where \( \{w(t)\} \) is a zero-mean complex WGN process with PSD \( N_0 \), \( g(t) \) is a unit energy signal, and the
   channel gain \( a \) is random with probability mass function
   \[
   P\{a = 0\} = 0.1 \quad \text{and} \quad P\{a = 2\} = 0.9.
   \]
(a) Determine the average probability of error $P_e$ for MPE detection. 
(b) What value does $P_e$ approach as $E/N_0$ approaches infinity? 
(c) Suppose the same signal is transmitted on two statistically independent channels with gains $a_1$ and $a_2$, where 
$$P\{a_1 = 0\} = P\{a_2 = 0\} = 0.1 \quad \text{and} \quad P\{a_1 = 2\} = P\{a_2 = 2\} = 0.9.$$ 
The additive noises on the two channels are also independent and identically distributed. The demodulator employs a matched filter for each channel and adds the two filter outputs to form the decision variable (which is compared to 0 for decision-making). Determine $P_e$ in this case. 
(d) For the case in part (c), what value does $P_e$ approach when $E/N_0$ approaches infinity? 

10. **Optimality of maximal-ratio combining** 
Consider BPSK signaling on an $L$-th order diversity channel. Each channel introduces a fixed attenuation and phase shift so that the received signal at the output of the $\ell$-th channel is: 
$$r_\ell(t) = \pm \alpha_\ell e^{j\phi_\ell} \sqrt{E} g_\ell(t) + w_\ell(t)$$ 
where the processes $w_\ell(t)$ are independent complex WGN processes with PSD $N_0$. 
The receiver uses the decision statistic 
$$R = \sum_{\ell=1}^L \beta_\ell \langle r_\ell, g_\ell \rangle$$ 
where the $\{\beta_\ell\}$ are complex weighting factors to be determined. A decision in favor of $+1$ ("symbol 0") is made if $r_\ell > 0$ and $-1$ ("symbol 0") otherwise. 
(a) Determine the p.d.f. of $R$ when $+1$ is transmitted. 
(b) Show that the probability of bit error $P_b$ is given by: 
$$P_b = Q \left( \sqrt{\frac{2E}{N_0}} \sum_{\ell=1}^L \frac{\text{Re}\{\beta_\ell \alpha_\ell e^{j\phi_\ell}\}}{\sqrt{\sum_{\ell=1}^L |\beta_\ell|^2}} \right).$$ 
(c) Determine the values of $\{\beta_\ell\}$ that minimize $P_b$. 
Hint: Use the Cauchy-Schwarz inequality