University of Illinois **Fall 2016**

ECE 562: Problem Set 6
Linear Equalizers, Signaling on Fading Channels

Due: Thursday, November 17 in class (**only turn in solutions to Problems 1-7**)
Reading: Lecture Notes 16-19, channel modeling slides

1. **[MSE of linear equalizers]**
The MSE for the k-th symbol when a linear equalizer $\underline{c}_k$ is used was defined in class to be:

$$\text{MSE}_k = \mathsf{E}\left[|\underline{c}_k^\dagger \underline{Z} - s_{m_k}|^2\right]$$

Using the fact that

$$\underline{Z} = \underline{h}_k s_{m_k} + \sum_{j \neq k} \underline{h}_j s_{m_j} + \underline{W}$$

show that

$$\text{MSE}_k = \mathcal{E}_\text{s}|\underline{c}_k^\dagger\ \underline{h}_k - 1|^2 + \mathcal{E}_\text{s}\sum_{j \neq k}|\underline{c}_k^\dagger\ \underline{h}_j|^2 + N_0\|\underline{c}_k\|^2$$

2. **[Minimum MSE]**
Consider the MMSE equalizer $\underline{c}_{k,\text{MMSE}} = \mathcal{E}_\text{s}(\mathcal{E}_\text{s} H\, H^\dagger + N_0 I)^{-1}\underline{h}_k$. Show that MSE achieved by this equalizer (which is the minimum MSE) is given by:

$$\text{MMSE}_k = \mathcal{E}_\text{s} - \mathcal{E}_\text{s}^2\ \underline{h}_k^\dagger(\mathcal{E}_\text{s} H\, H^\dagger + N_0 I)^{-1}\underline{h}_k$$

3. **[MMSE equallizer as SNR goes to infinity]**
As we discussed in class, the MMSE equalizer becomes ill-conditioned if we set $N_0 = 0$. Interestingly, however, we can still show that the MMSE equalizer converges to the ZF equalizer in limit as $N_0 \to 0$.

   (a) Show that MSE_k for $\underline{c}_{k,\text{ZF}}$ is given by:

   $$\text{MSE}_k(\underline{c}_{k,\text{ZF}}) = N_0\|\underline{c}_{k,\text{ZF}}\|^2.$$

   (b) Using part (a) argue that, for a fixed $\mathcal{E}_\text{s}$, MSE_k for $\underline{c}_{k,\text{MMSE}}$ satisfies:

   $$\lim_{N_0 \to 0} \text{MSE}_k(\underline{c}_{k,\text{MMSE}}) = 0.$$

   (c) Now use the results of part (b) and Problem 2 to conclude that the MMSE equalizer indeed converges to the ZF solution as $N_0 \to 0$, for fixed $\mathcal{E}_\text{s}$.

4. **[MMSE equalizer maximizes SINR]**
We showed in class that the SINR for the k-th symbol when a linear equalizer $\underline{c}_k$ is used is given by:

$$\text{SINR}_k = \frac{\mathcal{E}_\text{s}|\underline{c}_k^\dagger\ \underline{h}_k|^2}{\mathcal{E}_\text{s}\sum_{j \neq k}|\underline{c}_k^\dagger\ \underline{h}_j|^2 + N_0\|\underline{c}_k\|^2}$$

Using the result of Problem 2, show that $\underline{c}_{k,\text{MMSE}}$ maximizes SINR_k.
Hint: Consider the problem of maximizing SINR_k subject to $\underline{c}_k^\dagger\ \underline{h}_k = \alpha$, and show that the achieved maximum is independent of α.

5. **[Maximum SINR and Minimum MSE]**
Show that the maximum value of SINR_k is given by

$$\mathrm{SINR}_{k,\max} = \frac{\mathcal{E}_\mathrm{s}}{\mathrm{MMSE}_k} - 1$$

6. **[True or False]**
Determine if the following statements are True or False. You need to provide a brief justification for your answer to get credit.

   (a) If the mobile speed is 72 km/hr, the carrier frequency is 900 MHz, and symbol rate for communication is 10 symbols a second, then the mobile experiences slow fading.

   (b) In wireless communication, if the speed of the mobile is doubled the delay spread is also doubled.

   (c) A wireless channel with $\tau_{\mathrm{ds}} = 10^{-4}$ seconds and $f_{\max} = 100$ Hz operating with a bandwidth of 1.25 MHz is a flat fading channel.

   (d) For a fixed $\overline{\gamma}_\mathrm{b}$, $\overline{\mathrm{P}}_\mathrm{b}$ for binary DPSK signaling over a slow flat fading channel is always larger than $\overline{\mathrm{P}}_\mathrm{b}$ without fading, irrespective of the distribution of the fading.

7. **[Useful Result]**
Show that

$$\int_0^\infty Q(\sqrt{x}) \frac{e^{-x/\gamma}}{\gamma} dx = \frac{1}{2}\left(1 - \sqrt{\frac{\gamma}{2+\gamma}}\right).$$

Hint: Start with:

$$\int_{x=0}^\infty Q(\sqrt{x}) \frac{e^{-x/\gamma}}{\gamma} dx = \int_{x=0}^\infty \int_{t=\sqrt{x}}^\infty \frac{e^{-t^2/2}}{2\pi} \frac{e^{-x/\gamma}}{\gamma}\, dt\, dx = \int_{t=0}^\infty \int_{x=0}^{t^2} \frac{e^{-t^2/2}}{2\pi} \frac{e^{-x/\gamma}}{\gamma}\, dx\, dt$$

and use the fact that the Gaussian pdf integrates to 1 to conclude the result.

You do not need to turn in the solutions to Problems 8-10. I'll give you the solutions to these next Thursday when you turn in the solutions to Problems 1-7.

8. **[MPSK Signaling in Rayleigh Fading]**
For MPSK signaling,

$$\mathrm{P}_\mathrm{e}(\gamma_\mathrm{s}) = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \exp\left[-\frac{\gamma_\mathrm{s}\sin^2(\pi/M)}{\sin^2\theta}\right] d\theta\,.$$

Using this expression how that the average symbol error probability $\overline{\mathrm{P}}_\mathrm{e}$ for MPSK signaling in Rayleigh fading is given in closed form by

$$\overline{\mathrm{P}}_\mathrm{e} = \left(1 - \frac{1}{M}\right) - \frac{1}{\sqrt{1+a^2}}\left[\frac{1}{2} + \frac{1}{\pi}\tan^{-1}\left(\frac{\cot\pi/M}{\sqrt{1+a^2}}\right)\right],$$

where $a^2 = \frac{1}{\overline{\gamma}_\mathrm{s}\sin^2\pi/M}$.

Hint: You may need to use the following integral

$$\int_{\theta_1}^{\theta_2} \frac{1}{\mathrm{cosec}^2\theta + a^2} d\theta = \frac{1}{a^2}\left[\frac{1}{\sqrt{1+a^2}}\tan^{-1}\left(\frac{\cot\theta}{\sqrt{1+a^2}}\right) - \left(\frac{\pi}{2} - \theta\right)\right]_{\theta_1}^{\theta_2} \quad \text{for } 0 \le \theta_1 \le \theta_2 \le \pi/2.$$

9. **[Diversity]**
Consider BPSK with channel gain a, i.e., the received signal is

$$r(t) = \pm a\sqrt{\mathcal{E}}g(t) + w(t), \quad 0 \le t \le T,$$

where $\{w(t)\}$ is a zero-mean complex WGN process with PSD N_0, $g(t)$ is a unit energy signal, and the channel gain a is random with probability mass function

$$\mathrm{P}\{a = 0\} = 0.1 \quad \text{and} \quad \mathrm{P}\{a = 2\} = 0.9.$$

(a) Determine the average probability of error $\overline{\mathrm{P}}_\mathrm{e}$ for MPE detection.

(b) What value does $\overline{\mathrm{P}}_\mathrm{e}$ approach as $\mathcal{E}/N_0$ approaches infinity?

(c) Suppose the same signal is transmitted on two statistically *independent* channels with gains a_1 and a_2, where

$$\mathrm{P}\{a_1 = 0\} = \mathrm{P}\{a_2 = 0\} = 0.1 \quad \text{and} \quad \mathrm{P}\{a_1 = 2\} = \mathrm{P}\{a_2 = 2\} = 0.9.$$

The additive noises on the two channels are also independent and identically distributed. The demodulator employs a matched filter for each channel and adds the two filter outputs to form the decision variable (which is compared to 0 for decision-making). Determine $\overline{\mathrm{P}}_\mathrm{e}$ in this case.

(d) For the case in part (c), what value does $\overline{\mathrm{P}}_\mathrm{e}$ approach when $\mathcal{E}/N_0$ approaches infinity?

10. **[Optimality of maximal-ratio combining]**
Consider BPSK signaling on an L-th order diversity channel. Each channel introduces a fixed attenuation and phase shift so that the received signal at the output of the ℓ-th channel is:

$$r_\ell(t) = \pm \alpha_\ell e^{j\phi_\ell} \sqrt{\mathcal{E}}\, g_\ell(t) + w_\ell(t)$$

where the processes $w_\ell(t)$ are independent complex WGN processes with PSD N_0.

The receiver uses the decision statistic

$$R = \sum_{\ell=1}^{L} \beta_\ell \, \langle r_\ell, g_\ell \rangle$$

where the $\{\beta_\ell\}$ are complex weighting factors to be determined. A decision in favor of $+1$ ("symbol 0") is made if $r_I > 0$ and -1 ("symbol 0") otherwise.

(a) Determine the p.d.f. of R_I when $+1$ is transmitted.

(b) Show that the probability of bit error P_b is given by:

$$\mathrm{P}_\mathrm{b} = Q\left(\sqrt{\frac{2\mathcal{E}}{N_0}} \;\; \frac{\sum_{\ell=1}^{L} \mathrm{Re}\{\beta_\ell \alpha_\ell e^{j\phi_\ell}\}}{\sqrt{\sum_{\ell=1}^{L} |\beta_\ell|^2}} \right).$$

(c) Determine the values of $\{\beta_\ell\}$ that minimize P_b.
Hint: Use the Cauchy-Schwarz inequality