University of Illinois **Fall 2016**

ECE 562: Problem Set 3
Probability of Error for Linear Modulation, Orthogonal Modulation

Due: Tuesday October 4 in class
Reading: Lecture Notes 8-10; Sections 4.1 and 5.4 of Proakis & Salehi; Chapter 4 of Madhow

Reminder: Exam 1 will be held on Monday, October 10 from 7-8:30 PM in 2013 ECEB. It will cover the material for HW1-HW3. You will be allowed one sheet of notes (8.5" × 11", both sides) for the exam. Otherwise the exam is closed book.

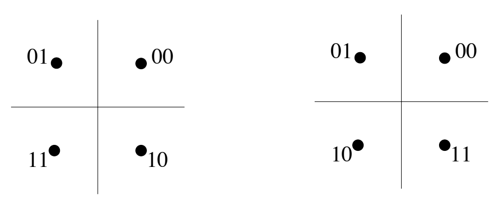
1. **[Performance of MPSK]**

   (a) Using the Intelligent Union Bound, show that the symbol error probability for MPSK signaling in AWGN is bounded by
   $$\mathrm{P_e} \leq 2Q\left(\sqrt{\frac{2\mathcal{E}_\mathrm{s}}{N_0}}\sin\frac{\pi}{M}\right).$$

   (b) Now, derive the following exact expression for $\mathrm{P_e}$.
   $$\mathrm{P_e} = \frac{1}{\pi}\int_0^{(M-1)\pi/M} \exp\left[-\frac{\mathcal{E}_\mathrm{s}}{N_0}\,\frac{\sin^2(\pi/M)}{\sin^2\theta}\right] d\theta\,.$$

   Hint: One way to proceed is to shift the origin to the signal point under consideration and use polar co-ordinates with the appropriate limits of integration.

2. **[Gray coding for QPSK]**
   Consider the following two bit assignments for QPSK

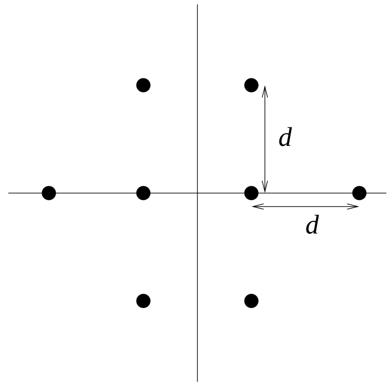


(1) (2)

   (a) Show that assignment (1), which corresponds to Gray coding, results in an average bit error probability of $\mathrm{P_b} = Q(\sqrt{2\gamma_\mathrm{b}})$.

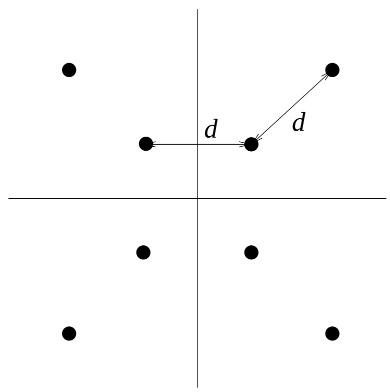
   (b) Show that under assignment (2), the first bit (from the left) sees an average probability of error of $Q(\sqrt{2\gamma_\mathrm{b}})$, whereas the second bit sees an average probability of error of $2Q(\sqrt{2\gamma_\mathrm{b}})[1 - Q(\sqrt{2\gamma_\mathrm{b}})]$. Thus
   $$\mathrm{P_b} = \frac{1}{2}Q(\sqrt{2\gamma_\mathrm{b}}) + Q(\sqrt{2\gamma_\mathrm{b}})[1 - Q(\sqrt{2\gamma_\mathrm{b}})] \;\geq\; Q(\sqrt{2\gamma_\mathrm{b}})$$

3. **[NNA versus IUB]**
   Consider the 8-ary constellation shown below:



(a) Assuming equally likely symbols, carefully draw the MPE decision regions for this constellation
(b) Find the NNA for $\mathrm{P_e}$ in terms of γ_s.
(c) Find the IUB for $\mathrm{P_e}$ in terms of γ_s.
(d) Find an exact expression for $\max_m \mathrm{P}_{\mathrm{e},m}$, the largest conditional symbol error probability.

4. **[Gray Coding and Bit Error Probability]**
Consider the 8-ary QAM constellation shown below (where all nearest neighbors are equidistant):



(a) Determine whether you can label the signal points using three bits so that nearest neighbors differ by at most one bit (Gray coding). If so, find such a labeling. If not, state why not and find a labeling that minimizes the maximum number of bit transitions between neighbors.
(b) For the labelings found in part (a), compute the nearest neighbor approximation for the average bit error probability $\mathrm{P_b}$ as a function of d^2/N_0.

5. **["Semi-Orthogonal" Signal Set]**
Consider the signal set with $M = 2N$ signals given by:

$$s_m(t) = \begin{cases} \sqrt{\mathcal{E}} g_m(t) & m = 0, 1, \ldots, N-1 \\ j\sqrt{\mathcal{E}} g_{m-N}(t) & m = N, \ldots, M-1. \end{cases}$$

where $\{g_k(t)\}_{k=1}^{N}$ are **real-valued** orthonormal functions. Clearly this signal set satisfies: $\mathsf{Re}[\rho_{k,\ell}] = 0$, for $k \neq \ell$, but *not* $\rho_{k,\ell} = 0$, for $k \neq \ell$

(a) Argue that $R_k = \langle r(t), g_k(t) \rangle$, $k = 0, 1, \ldots, N-1$, form sufficient statistics for optimal decision making at the receiver for an AWGN channel.
(b) Now define the M real-valued statistics

$$y_m = \begin{cases} r_{m,I} & m = 0, 1, \ldots, N-1 \\ r_{(m-N),Q} & m = N, \ldots, M-1. \end{cases}$$

Show that the MPE decision rule is given by

$$\hat{m}_{\text{MPE}} = \arg\max_{m} y_m$$

(c) Find an expression for P_e for the MPE decision rule.

6. **[Asymptotic Performance of Orthogonal Signaling]**
In this problem you will show the following result for M-ary orthogonal modulation

$$\lim_{M\to\infty} \text{P}_\text{c} = \begin{cases} 1 & \gamma_\text{b} > \ln 2 \\ 0 & \gamma_\text{b} < \ln 2 \end{cases}$$

where P_c is the probability of correct decision making.

Recall that we showed in class that

$$\text{P}_\text{c} = \int_{-\infty}^{\infty} \left[1 - Q(x + \sqrt{2\gamma_\text{b} \log_2 M}\,)\right]^{M-1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$$

(a) Show that for any x,

$$\lim_{M\to\infty} \left[1 - Q(x + \sqrt{2\gamma_\text{b} \log_2 M}\,)\right]^{M-1} = \begin{cases} 1 & \gamma_\text{b} > \ln 2 \\ 0 & \gamma_\text{b} < \ln 2 \end{cases}$$

Hint: Use L'Hopital's rule on the log of the expression before taking the limit.

(b) Use the result of part (a) to arrive at the desired result.