ECE 562: Problem Set 3
Probability of Error for Linear Modulation, Orthogonal Modulation

Due: Tuesday October 4 in class
Reading: Lecture Notes 8-10; Sections 4.1 and 5.4 of Proakis & Salehi; Chapter 4 of Madhow

Reminder: Exam 1 will be held on Monday, October 10 from 7-8:30 PM in 2013 ECEB. It will cover the material for HW1-HW3. You will be allowed one sheet of notes (8.5” × 11”, both sides) for the exam. Otherwise the exam is closed book.

1. [Performance of MPSK]

(a) Using the Intelligent Union Bound, show that the symbol error probability for MPSK signaling in AWGN is bounded by

\[ P_e \leq 2Q\left(\sqrt{\frac{2E_s}{N_0}} \sin \frac{\pi}{M}\right). \]

(b) Now, derive the following exact expression for \( P_e \).

\[ P_e = \frac{1}{\pi} \int_0^{(M-1)\pi/M} \exp\left[-\frac{E_s}{N_0} \frac{\sin^2(\pi/M)}{\sin^2 \theta}\right] d\theta. \]

Hint: One way to proceed is to shift the origin to the signal point under consideration and use polar co-ordinates with the appropriate limits of integration.

2. [Gray coding for QPSK]

Consider the following two bit assignments for QPSK

(a) Show that assignment (1), which corresponds to Gray coding, results in an average bit error probability of \( P_b = Q(\sqrt{2\gamma_b}) \).

(b) Show that under assignment (2), the first bit (from the left) sees an average probability of error of \( Q(\sqrt{2\gamma_b}) \), whereas the second bit sees an average probability of error of \( 2Q(\sqrt{2\gamma_b})[1 - Q(\sqrt{2\gamma_b})] \).

Thus

\[ P_b = \frac{1}{2}Q(\sqrt{2\gamma_b}) + Q(\sqrt{2\gamma_b})[1 - Q(\sqrt{2\gamma_b})] \geq Q(\sqrt{2\gamma_b}) \]

3. [NNA versus IUB]

Consider the 8-ary constellation shown below:
(a) Assuming equally likely symbols, carefully draw the MPE decision regions for this constellation.
(b) Find the NNA for $P_e$ in terms of $\gamma_s$.
(c) Find the IUB for $P_e$ in terms of $\gamma_s$.
(d) Find an exact expression for $\max_m P_{e,m}$, the largest conditional symbol error probability.

4. **[Gray Coding and Bit Error Probability]**
Consider the 8-ary QAM constellation shown below (where all nearest neighbors are equidistant):

![Gray Coding Diagram](gray-coding-diagram.png)

(a) Determine whether you can label the signal points using three bits so that nearest neighbors differ by at most one bit (Gray coding). If so, find such a labeling. If not, state why not and find a labeling that minimizes the maximum number of bit transitions between neighbors.
(b) For the labelings found in part (a), compute the nearest neighbor approximation for the average bit error probability $P_b$ as a function of $d^2/N_0$.

5. **[“Semi-Orthogonal” Signal Set]**
Consider the signal set with $M = 2N$ signals given by:

$$s_m(t) = \begin{cases} \sqrt{E}g_m(t) & m = 0, 1, \ldots, N - 1 \\ j\sqrt{E}g_{m-N}(t) & m = N, \ldots, M - 1. \end{cases}$$

where $\{g_k(t)\}_{k=1}^N$ are real-valued orthonormal functions. Clearly this signal set satisfies: $\text{Re}[\rho_{k,\ell}] = 0$, for $k \neq \ell$, but not $\rho_{k,\ell} = 0$, for $k \neq \ell$

(a) Argue that $R_k = \langle r(t), g_k(t) \rangle$, $k = 0, 1, \ldots, N - 1$, form sufficient statistics for optimal decision making at the receiver for an AWGN channel.
(b) Now define the $M$ real-valued statistics:

$$y_m = \begin{cases} r_{m,I} & m = 0, 1, \ldots, N - 1 \\ r_{(m-N),Q}/2 & m = N, \ldots, M - 1. \end{cases}$$
Show that the MPE decision rule is given by
\[ \hat{m}_{\text{MPE}} = \arg \max_m y_m \]

(c) Find an expression for \( P_c \) for the MPE decision rule.

6. [Asymptotic Performance of Orthogonal Signaling]
In this problem you will show the following result for \( M \)-ary orthogonal modulation
\[ \lim_{M \to \infty} P_c = \begin{cases} 1 & \gamma_b > \ln 2 \\ 0 & \gamma_b < \ln 2 \end{cases} \]
where \( P_c \) is the probability of correct decision making.
Recall that we showed in class that
\[ P_c = \int_{-\infty}^{\infty} \left[ 1 - Q(x + \sqrt{2\gamma_b \log_2 M}) \right]^{M-1} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx \]
(a) Show that for any \( x \),
\[ \lim_{M \to \infty} \left[ 1 - Q(x + \sqrt{2\gamma_b \log_2 M}) \right]^{M-1} = \begin{cases} 1 & \gamma_b > \ln 2 \\ 0 & \gamma_b < \ln 2 \end{cases} \]
Hint: Use L’Hopital’s rule on the log of the expression before taking the limit.
(b) Use the result of part (a) to arrive at the desired result.