ECE 562: Problem Set 1
Random Processes, White Gaussian Noise, Complex Baseband, Signal
Space Concepts

Due: Tuesday September 6 in class
Reading: Lecture Notes 1-4, Chapters 1-3 of Wozencraft & Jacobs, Sections 2.2, 3.1, and 3.3 of Madhow.

1. [Random process with exponential sample paths]
Suppose $X$ is a random variable that is uniformly distributed on the interval $[0, 1]$, that is $f_X(x)$ is 1 on the interval $[0, 1]$ and 0 otherwise.

(a) Suppose that a random process (which is only defined for $t > 0$) is given by $Y(t) = e^{-tX}$. Find the cdf and pdf of the random variable $Y(t_0)$, where $t_0$ is a fixed positive number.
(b) Find the mean and autocorrelation function of the random process $\{Y(t)\}$.
(c) Is $\{Y(t)\}$ WSS?

2. [Signaling in AWGN]
Consider the signal $s(t) = \sin(\pi t/T)\mathbb{1}_{0 \leq t \leq T}$. Suppose this signal is corrupted by AWGN with two-sided PSD $N_0/2$ to form the received signal:

$r(t) = s(t) + n(t)$

Further suppose that we form the random variables $Y_1$ and $Y_2$ as:

$Y_1 = \int_0^T r(t) \sin(\pi t/T) dt, \quad Y_2 = \int_0^T r(t) \cos(\pi t/T) dt$

Express the following probabilities in terms of $T$, $N_0$, $\gamma$ and the $Q$ function:

(a) $P\{Y_1 > \gamma\}$
(b) $P\{Y_2 > \gamma\}$
(c) $P\{Y_1 + Y_2 > \gamma\}$

3. [Passband to baseband]

(a) Show that the following scheme works for converting signals from passband to complex baseband:

\[ \begin{array}{c}
\bigotimes \quad \text{LPF} \\
\sqrt{2} \cos 2\pi f_c t \\
\bigotimes \quad \text{LPF} \\
-\sqrt{2} \sin 2\pi f_c t
\end{array} \]

\[ \begin{array}{c}
\ddot{s}(t) \\
\dddot{s}(t)
\end{array} \]

\[ \begin{array}{c}
s_I(t) \\
s_Q(t)
\end{array} \]
(b) Find the complex baseband representation of the real-valued passband signal:

\[ \tilde{s}(t) = g(t) \cos(2\pi f_c t + \frac{\pi}{4}) + f(t) \sin(2\pi f_c t + \frac{\pi}{4}) \]

where \( g(t) \) and \( f(t) \) are real-valued signals with bandwidth \( W \) that is much smaller than \( f_c \).

4. **[Baseband representation of channel]**
   Show that the complex baseband representation of the passband channel satisfies:

\[ H(f) = \frac{1}{\sqrt{2}} \tilde{H}(f + f_c) \]

5. **[Complex Gaussian Preview]**
   Suppose \( X_I \) and \( X_Q \) are independent \( \mathcal{N}(0,1) \) random variables and we create \( Y_I \) and \( Y_Q \) using the transformation:

\[ Y_I + jY_Q = (X_I + jX_Q)e^{j\phi} \]

for some \( \phi \in [0, 2\pi] \). Show that \((Y_I, Y_Q)\) have the same joint pdf as \((X_I, X_Q)\).

6. **[Signal Space]**
   Consider the following three waveforms:

   ![Graph of three waveforms](image)

   (a) Show that these waveforms are orthonormal.
   (b) Express the following waveform \( s(t) \) as a linear combination of \( f_n(t), \ n = 1, 2, 3 \).
7. [Gram-Schmidt]
Determine an orthonormal basis for the span of the following four signals:

8. [Signal Spaces as Vector Spaces]
Let \( \{f_\ell(t)\}_{\ell=1}^n \) be an orthonormal basis for \( S \)

(a) Suppose \( s(t) \in S \) has the representation in terms of this basis as:

\[
s(t) = \sum_{\ell=1}^n s_\ell f_\ell(t)
\]

and define the column vector \( \mathbf{s} = [s_1 \ s_2 \ldots s_n]^\top \). Show that

\[
\|s\| = \sqrt{\sum_{\ell=1}^n |s_\ell|^2} = \sqrt{s^\dagger s} = \|\mathbf{s}\|
\]

(b) Now suppose \( s_k(t), s_m(t) \in S \) and let

\[
\mathbf{s}_k = [s_{k,1} \ s_{k,2} \ldots s_{k,n}]^\top \quad \mathbf{s}_m = [s_{m,1} \ s_{m,2} \ldots s_{m,n}]^\top.
\]

Show that

\[
\langle s_k, s_m \rangle = \sum_{\ell=1}^n s_{k,\ell} s_{m,\ell} = \mathbf{s}_m^\dagger \mathbf{s}_k = \langle \mathbf{s}_k, \mathbf{s}_m \rangle
\]

(c) Show that the distance \( d_{km} \) between two signals \( s_k(t) \) and \( s_m(t) \) is given in terms of their energies and correlation coefficient as:

\[
d_{km} = \left( \mathcal{E}_k + \mathcal{E}_m - 2\sqrt{\mathcal{E}_k \mathcal{E}_m \text{Re}[\rho_{k,m}]} \right)^{\frac{1}{2}}
\]