University of Illinois **Fall 2016**

ECE 562: Problem Set 1
Random Processes, White Gaussian Noise, Complex Baseband, Signal Space Concepts

Due: Tuesday September 6 in class
Reading: Lecture Notes 1-4, Chapters 1-3 of Wozencraft & Jacobs, Sections 2.2, 3.1, and 3.3 of Madhow.

1. **[Random process with exponential sample paths]**
   Suppose X is a random variable that is uniformly distributed on the interval $[0, 1]$, that is $f_X(x)$ is 1 on the interval $[0, 1]$ and 0 otherwise.

   (a) Suppose that a random process (which is only defined for $t > 0$) is given by $Y(t) = e^{-tX}$. Find the cdf and pdf of the random variable $Y(t_0)$, where t_0 is a fixed positive number.

   (b) Find the mean and autocorrelation function of the random process $\{Y(t)\}$.

   (c) Is $\{Y(t)\}$ WSS?

2. **[Signaling in AWGN]**
   Consider the signal $s(t) = \sin(\pi t/T)\mathbb{1}_{\{0 \le t \le T\}}$. Suppose this signal is corrupted by AWGN with two-sided PSD $N_0/2$ to form the received signal:

$$r(t) = s(t) + n(t)$$

   Further suppose that we form the random variables Y_1 and Y_2 as:

$$Y_1 = \int_0^T r(t)\sin(\pi t/T)dt, \qquad Y_2 = \int_0^T r(t)\cos(\pi t/T)dt$$

   Express the following probabilities in terms of T, N_0, γ and the Q function:

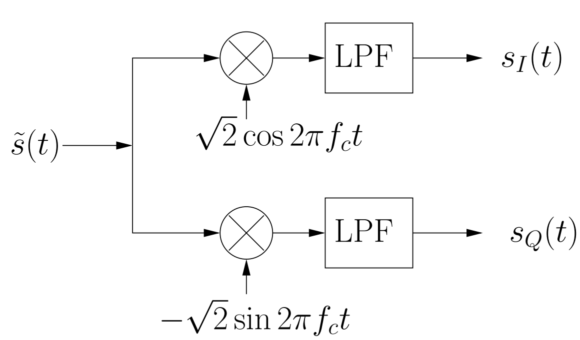
   (a) $\mathsf{P}\{Y_1 > \gamma\}$

   (b) $\mathsf{P}\{Y_2 > \gamma\}$

   (c) $\mathsf{P}\{Y_1 + Y_2 > \gamma\}$

3. **[Passband to baseband]**

   (a) Show that the following scheme works for converting signals from passband to complex baseband:



(b) Find the complex baseband representation of the real-valued passband signal:

$$\tilde{s}(t) = g(t)\cos(2\pi f_c t + \tfrac{\pi}{4}) + f(t)\sin(2\pi f_c t + \tfrac{\pi}{4})$$

where $g(t)$ and $f(t)$ are real-valued signals with bandwidth W that is much smaller than f_c.

4. **[Baseband representation of channel]**
Show that the complex baseband representation of the passband channel satisfies:

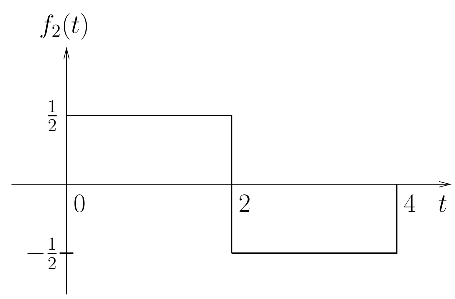
$$H(f) = \frac{1}{\sqrt{2}}\tilde{H}_+(f + f_c)$$

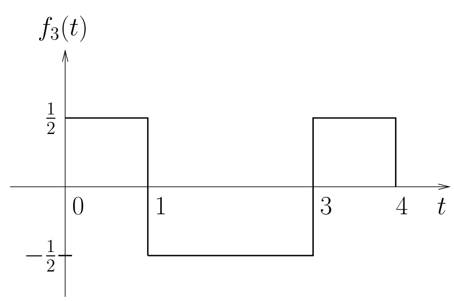
5. **[Complex Gaussian Preview]**
Suppose X_I and X_Q are independent $\mathcal{N}(0,1)$ random variables and we create Y_I and Y_Q using the transformation:

$$Y_I + jY_Q = (X_I + jX_Q)e^{j\phi}$$

for some $\phi \in [0, 2\pi]$. Show that (Y_I, Y_Q) have the same joint pdf as (X_I, X_Q).

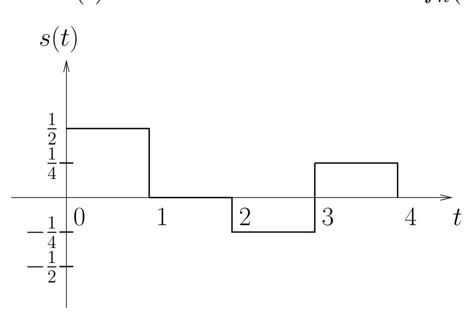
6. **[Signal Space]**
Consider the following three waveforms:



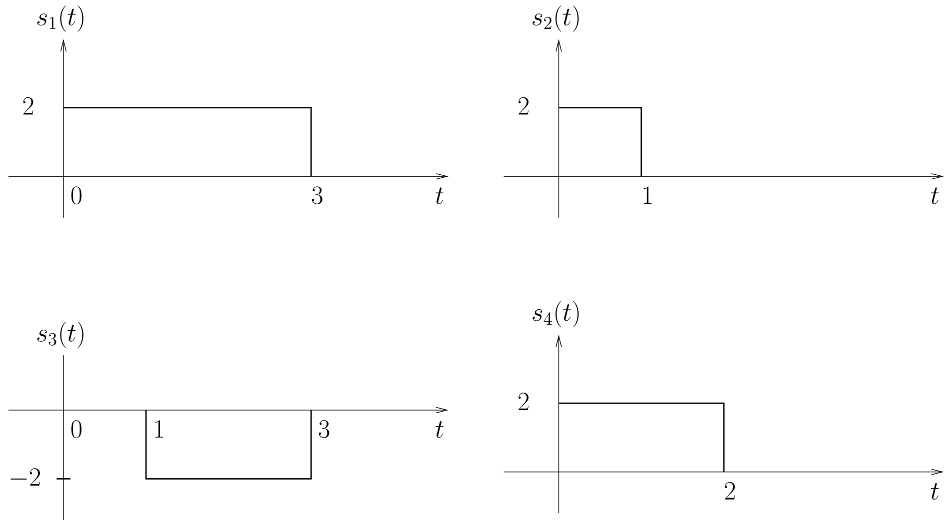


(a) Show that these waveforms are orthonormal.

(b) Express the following waveform $s(t)$ as a linear combination of $f_n(t),\ n = 1, 2, 3$.



7. **[Gram-Schmidt]**
Determine an orthonormal basis for the span of the following four signals:



8. **[Signal Spaces as Vector Spaces]**
Let $\{f_\ell(t)\}_{\ell=1}^n$ be an orthonormal basis for $\mathcal{S}$

(a) Suppose $s(t) \in \mathcal{S}$ has the representation in terms of this basis as:

$$s(t) = \sum_{\ell=1}^{n} s_\ell \; f_\ell(t)$$

and define the column vector $\underline{s} = [s_1 \; s_2 \cdots s_n]^\top$. Show that

$$\|s\| = \sqrt{\sum_{\ell=1}^{n} |s_\ell|^2} = \sqrt{\underline{s}^\dagger \underline{s}} = \|\underline{s}\|$$

(b) Now suppose $s_k(t), s_m(t) \in \mathcal{S}$ and let

$$\underline{s}_k = [s_{k,1} \; s_{k,2} \cdots s_{k,n}]^\top \qquad \underline{s}_m = [s_{m,1} \; s_{m,2} \cdots s_{m,n}]^\top.$$

Show that

$$\langle s_k, s_m \rangle \;=\; \sum_{\ell=1}^{n} s_{k,\ell} s_{m,\ell}^* \;=\; \underline{s}_m^\dagger \underline{s}_k \;=\; \langle \underline{s}_k, \underline{s}_m \rangle$$

(c) Show that the distance d_{km} between two signals $s_k(t)$ and $s_m(t)$ is given in terms of their energies and correlation coefficient as:

$$d_{km} = \left(\mathcal{E}_k + \mathcal{E}_m - 2\sqrt{\mathcal{E}_k \mathcal{E}_m} \mathrm{Re}[\rho_{k,m}] \right)^{\frac{1}{2}}$$