ECE 561 **Spring 2017**

March 8, 2017

EXAM 1: 7:00–8:30 PM (*50 points total*)

1. **[Simple Hypothesis Testing (20 pts)]**
   Consider the binary hypothesis testing problem with

$$p_1(y) = \frac{1}{2}e^{-\frac{y}{2}}\,\mathbb{I}_{\{y\geq 0\}}, \text{ and } p_0(y) = e^{-y}\,\mathbb{I}\{y \geq 0\}$$

   (a) Find a Bayes rule for uniform costs and equal priors.
   (b) Find the Bayes risk for the Bayes rule of part (a)
   (c) Find a Neyman-Pearson rule for level α, for $\alpha \in (0, 1)$.
   (d) Find the probability of detection as a function of α for the N-P rule of part (c).
   (e) Find a minimax rule for uniform costs. (You need to solve a quadratic equation for the threshold.)
   (f) Find the minimax risk for the minimax rule of part (e).

2. **[Composite Hypothesis Testing (15 points)]**
   Consider the hypothesis testing problem:

$$H_0 : \boldsymbol{Y} = \boldsymbol{Z}$$

$$H_1 : \boldsymbol{Y} = \boldsymbol{s}(\theta) + \boldsymbol{Z}$$

   where θ is a deterministic but unknown parameter that takes values in the set $\{1, 2\}$, $\boldsymbol{Z} \sim \mathcal{N}(\mathbf{0}, \mathsf{I})$, and

$$\boldsymbol{s}(1) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad \boldsymbol{s}(2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}.$$

   (a) Is there a UMP test between H_0 and H_1? If so, find it (for a level of α). If not, explain why not.
   (b) Find an α-level GLRT for testing between H_0 and H_1.
   (c) Argue that the probability of detection for the GLRT of part (b) is not a function of θ, and then find the probability of detection as a function of α.

3. **[Shorts (15 points)]**
   (a) Suppose the minimum Bayes risk function for *uniform costs* for a binary hypothesis testing problem is given by:

$$V(\pi_0) = \begin{cases} \frac{\pi_0}{2} & \text{if } \pi_0 \in [0, 0.2) \\ \frac{\pi_0 + 0.1}{3} & \text{if } \pi_0 \in [0.2, 0.5) \\ \frac{2(1-\pi_0)}{5} & \text{if } \pi_0 \in [0.5, 1] \end{cases}$$

   Find the threshold and randomization constant for the minimax LRT, and the corresponding minimax risk.
   (b) Consider testing between $H_0 : \boldsymbol{Y} = \boldsymbol{Z}$ and $H_1 : \boldsymbol{Y} = \boldsymbol{s} + \boldsymbol{Z}$, where $\boldsymbol{Z} \sim \mathcal{N}(\mathbf{0}, \mathsf{I})$ and $\boldsymbol{s}$ is deterministic signal. We wish to design a linear detector with statistic $T(\boldsymbol{y}) = \boldsymbol{\mu}^\top \boldsymbol{y}$ for this problem using the deflection criterion. Find a value of $\boldsymbol{\mu}$ that maximizes the deflection.
   (c) Consider the problem of sequentially testing between the distributions

$$p_1(y_k) = \begin{cases} \frac{2}{3} & \text{if } y_k = 1 \\ \frac{1}{3} & \text{if } y_k = 0 \end{cases}, \quad p_0(y_k) = \begin{cases} \frac{1}{3} & \text{if } y_k = 1 \\ \frac{2}{3} & \text{if } y_k = 0 \end{cases}$$

   using an SPRT with thresholds $a = -10 \ln 2$ and $b = 10 \ln 2$ on the log-likelihood ratio.
   Find the error probabilities $\mathrm{P_F}$ and $\mathrm{P_M}$.