EXAM 1: 7:00–8:30 PM (50 points total)

1. [Simple Hypothesis Testing (20 pts)]
   Consider the binary hypothesis testing problem with
   \[ p_1(y) = \frac{1}{2} e^{-\frac{y^2}{2}} \mathbb{1}_{y \geq 0}, \quad p_0(y) = e^{-y} \mathbb{1}_{y \geq 0} \]
   (a) Find a Bayes rule for uniform costs and equal priors.
   (b) Find the Bayes risk for the Bayes rule of part (a)
   (c) Find a Neyman-Pearson rule for level \( \alpha \), for \( \alpha \in (0, 1) \).
   (d) Find the probability of detection as a function of \( \alpha \) for the N-P rule of part (c).
   (e) Find a minimax rule for uniform costs. (You need to solve a quadratic equation for the threshold.)
   (f) Find the minimax risk for the minimax rule of part (e).

2. [Composite Hypothesis Testing (15 points)]
   Consider the hypothesis testing problem:
   \[ H_0 : Y = Z \]
   \[ H_1 : Y = s(\theta) + Z \]
   where \( \theta \) is a deterministic but unknown parameter that takes values in the set \( \{1, 2\} \), \( Z \sim \mathcal{N}(0, 1) \), and
   \[ s(1) = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \quad s(2) = \begin{bmatrix} 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}. \]
   (a) Is there a UMP test between \( H_0 \) and \( H_1 \)? If so, find it (for a level of \( \alpha \)). If not, explain why not.
   (b) Find an \( \alpha \)-level GLRT for testing between \( H_0 \) and \( H_1 \).
   (c) Argue that the probability of detection for the GLRT of part (b) is not a function of \( \theta \), and then find the probability of detection as a function of \( \alpha \).

3. [Shorts (15 points)]
   (a) Suppose the minimum Bayes risk function for uniform costs for a binary hypothesis testing problem is given by:
   \[ V(\pi_0) = \begin{cases} \pi_0^2 & \text{if } \pi_0 \in [0, 0.2) \\ \pi_0 + 0.1 & \text{if } \pi_0 \in [0.2, 0.5) \\ \frac{2(1-\pi_0)}{3} & \text{if } \pi_0 \in [0.5, 1] \end{cases} \]
   Find the threshold and randomization constant for the minimax LRT, and the corresponding minimax risk.
   (b) Consider testing between \( H_0 : Y = Z \) and \( H_1 : Y = s + Z \), where \( Z \sim \mathcal{N}(0, 1) \) and \( s \) is deterministic signal. We wish to design a linear detector with statistic \( T(y) = \mu^\top y \) for this problem using the deflection criterion. Find a value of \( \mu \) that maximizes the deflection.
   (c) Consider the problem of sequentially testing between the distributions
   \[ p_1(y_k) = \begin{cases} \frac{2}{3} & \text{if } y_k = 1 \\ \frac{1}{3} & \text{if } y_k = 0 \end{cases}, \quad p_0(y_k) = \begin{cases} \frac{1}{3} & \text{if } y_k = 1 \\ \frac{2}{3} & \text{if } y_k = 0 \end{cases} \]
   using an SPRT with thresholds \( a = -10 \ln 2 \) and \( b = 10 \ln 2 \) on the log-likelihood ratio.
   Find the error probabilities \( P_F \) and \( P_M \).