EXAM 1: Solutions

1.  (a) The Bayes solution with uniform costs and equal priors is

\[ \delta_B(y) = \begin{cases} 
1 & \text{if } L(y) \geq 1 \\
0 & \text{if } L(y) < 1 
\end{cases} \]

Now, \( L(0) = 0, L(1) = 2/3, L(2) = 4/3, \) and \( L(3) = 2. \) Thus, the Bayes rule reduces to

\[ \delta_B(y) = \begin{cases} 
1 & \text{if } y = 2, 3 \\
0 & \text{if } y = 0, 1 
\end{cases} \]

Note that this is same as the rule \( \delta^{(1)}(y) \) of part (c).

(b) The Bayes risk is given by

\[ r(\delta_B) = 0.5 \, P_0\{Y \in \{2, 3\}\} + 0.5 \, P_1\{Y \in \{0, 1\}\} \]
\[ = 0.5 \left( 0.5 \right) + 0.5 \left( 1/6 \right) \]
\[ = 1/3. \]

(c) As noted above, \( \delta^{(1)} = \delta_B. \) Further, \( R_0(\delta^{(2)}) = P_0\{Y = 3\} = 1/4 \) and \( R_1(\delta^{(2)}) = P_1\{Y \neq 3\} = 1/2. \) Thus,

\[ r(\pi_0, \delta^{(1)}) = (1/2)\pi_0 + (1/6)(1 - \pi_0) \]
\[ r(\pi_0, \delta^{(2)}) = (1/4)\pi_0 + (1/2)(1 - \pi_0). \]

The risk lines are shown in Fig. 1.

![Figure 1: Risk lines for Prob 1 (c).](image-url)
(d) The risk lines in part (c) have \( \pi_0 = 4/7 \) as the point of intersection. For this prior, \( \frac{\pi_0}{1-\pi_0} = 4/3 \). Since, \( L(0) = 0, L(1) = 2/3, L(2) = 4/3, \) and \( L(3) = 2, \) both \( \delta^{(1)} \) and \( \delta^{(2)} \) are Bayes rules for this prior. The Bayes risk for this prior is equal to 5/14. An equalizer rule with risk equal to 5/14 will be a minimax rule. To obtain an equalizer rule, we randomize between the rules \( \delta^{(1)} \) and \( \delta^{(2)} \).

The randomization parameter can be obtained by solving for \( \lambda \) in
\[
\frac{5}{14} = \lambda \left( \frac{1}{2} \right) + \left( 1 - \lambda \right) \left( \frac{1}{4} \right).
\]
This gives \( \lambda = 3/7 \). Thus the minimax rule is
\[
\delta_m = \begin{cases} 
\delta^{(1)} & \text{w. p. } 3/7 \\
\delta^{(2)} & \text{w. p. } 4/7 
\end{cases} = \begin{cases} 
1 & \text{if } y = 3 \\
1 & \text{w. p. } 3/7 \text{ if } y = 2 \\
0 & \text{if } y = 0, 1
\end{cases}.
\]

See Fig. 1.

(e) Due to the monotone nature of \( L(y) \) as a function of \( y \) and the discrete nature of observations, any Neyman-Pearson rule is of the form
\[
\delta_{NP}(y) = \begin{cases} 
1 & \text{if } y > k \\
1 & \text{w. p. } \gamma \text{ if } y = k \\
0 & \text{if } y < k
\end{cases},
\]
for \( k = 0, 1, 2, 3 \). The threshold \( k \) and parameter \( \gamma \) is chosen to meet the \( P_F \) constraint of \( \alpha \) with equality. The complementary C.D.F. of \( Y \) under \( H_0 \) is given by
\[
P_0(Y > k) = 3/4 - 1/4 k \quad \text{for } k = 0, 1, 2, 3.
\]
Thus, for \( \alpha = 1/3 \), we have \( P_0(Y > 2) < \alpha < P_0(Y > 1) \). This gives threshold \( k = 2 \) and randomization parameter \( \gamma \) as
\[
\gamma = \frac{1/3 - 1/4}{1/4} = 1/3.
\]
Thus, the NP rule for \( \alpha = 1/3 \) is equal to
\[
\delta_{NP}(y) = \begin{cases} 
1 & \text{if } y = 3 \\
1 & \text{w. p. } 1/3 \text{ if } y = 2 \\
0 & \text{if } y = 0, 1
\end{cases}.
\]

(f) The NP test of part (e) has the following \( P_D \):
\[
P_D(\delta_{NP}) = P_1\{Y = 3\} + 1/3 P_1\{Y = 2\} = 11/18.
\]

2. (a) We find the NP test for \( H_0 \) vs \( H_1 \) for a fixed \( \theta > 0 \) and check if the rule is independent of \( \theta \).

There are no point masses (we are dealing with Gaussian distribution), so no randomization is needed. A NP rule is of the form
\[
\delta_{NP}(y) = \begin{cases} 
1 & \text{if } \ln L_\theta(y) \geq \eta \\
0 & \text{otherwise}
\end{cases}.
\]
The log likelihood can be simplified to
\[
\ln L_\theta(y) = y_1 \theta + y_2 \theta^2 - \theta^2/2 - \theta^4/2.
\]
Due to the presence of \(\theta^2\) as the coefficient of \(y_2\), the dependence on \(\theta\) cannot be removed.
If \(\theta = 1\), the NP rule reduces to
\[
\delta_{\text{NP}}(y) = \begin{cases} 
1 \quad &\text{if } y_1 + y_2 \geq \eta^{(1)} \\
0 \quad &\text{otherwise}
\end{cases}.
\]
If \(\theta = 2\), the NP rule reduces to
\[
\delta_{\text{NP}}(y) = \begin{cases} 
1 \quad &\text{if } y_1 + 2 y_2 \geq \eta^{(2)} \\
0 \quad &\text{otherwise}
\end{cases}.
\]
As a result no UMP solution exists.
(b) The test statistic for the LMP test is given by
\[
L_\ell(y) = \frac{\partial p_\theta(y)}{\partial \theta} \bigg|_{\theta=0}
\]
Now,
\[
\frac{\partial p_\theta(y)}{\partial \theta} = p_0(y)((y_1 - \theta) + 2\theta(y_2 - \theta^2))
\]
Thus,
\[
\frac{\partial p_\theta(y)}{\partial \theta} \bigg|_{\theta=0} = p_0(y)y_1,
\]
and
\[
L_\ell(y) = \frac{p_0(y)y_1}{p_0(y)} = y_1.
\]
Under \(H_0\), \(Y_1 \sim \mathcal{N}(0, 1)\). Thus,
\[
P_F = P_0\{Y_1 > \eta\} = \alpha
\]
imply \(\eta = Q^{-1}(\alpha)\), and the \(\alpha\)-level locally optimum test is given by
\[
\delta_{\text{LMP}}(y) = \begin{cases} 
1 \quad &\text{if } y_1 \geq Q^{-1}(\alpha) \\
0 \quad &\text{otherwise}
\end{cases}.
\]
(c) The \(P_D\) of the LMP test of part (b) for any fixed \(\theta > 0\) is given by
\[
P_D(\delta_{\text{LMP}}) = P_1(Y_1 > Q^{-1}(\alpha)) = Q(|Q^{-1}(\alpha) - \theta|).
\]

3. (a) Since \(V(\pi_0)\) is a concave function, the partial plot provided is enough to conclude that \(V(\pi_0)\) is maximized at \(\pi_0 = 0.5\). Moreover the costs are uniform and the risk curve is differentiable at \(\pi_0 = 0.5\). Thus, a minimax rule is the Bayes rule for this particular prior \(\pi_0 = 0.5\):
\[
\delta_m(y) = \begin{cases} 
1 \quad &\text{if } L(y) \geq 1 \\
0 \quad &\text{otherwise}
\end{cases}.
\]
The minimax risk is the risk of this rule which is 0.3.
(b) The log GLRT statistic is given by

\[
\ln L_G(y) = \ln \left( \sup_{\theta} p_{\theta}(y) \right) = \sup_{\theta} \ln L_{\theta}(y)
\]

\[
= \max\{\ln L_1(y), \ln L_{-1}(y)\}
\]

\[
= (y - 1/2)\mathbb{1}_{y \geq 0} + (-y - 1/2)\mathbb{1}_{y < 0}
\]

\[
= |y| - 1/2.
\]

Hence, the GLRT is given by

\[
\delta_{\text{GLRT}}(y) = \begin{cases} 
1 & \text{if } |y| \leq \eta \\
0 & \text{if } |y| > \eta 
\end{cases} \quad \text{for some } \eta.
\]

(c) **Solution A:** The Bayes rule with \(U\) as an observation is given by

\[
\delta_U^B(y) = \begin{cases} 
1 & \text{if } L_U(y) \geq \frac{\pi_0}{1-\pi_0} = \eta(\pi_0) \\
0 & \text{otherwise}
\end{cases}
\]

where, \(L_U(u)\) is the likelihood ratio of the observation \(U = u\) given by \(L_U(u) = \frac{p_1^U(u)}{p_0^U(u)}\), with \(p_1^U(u)\) and \(p_0^U(u)\) as the p.m.fs of \(U\) under \(H_1\) and \(H_0\), respectively. Now note that

\[
L_U(1) = \frac{p_1^U(1)}{p_0^U(1)} = \frac{P_1\{U = 1\}}{P_0\{U = 1\}} = \frac{P_1\{L(Y) \geq \eta(\pi_0)\}}{P_0\{L(Y) \geq \eta(\pi_0)\}} \geq \eta(\pi_0),
\]

and

\[
L_U(0) = \frac{p_1^U(0)}{p_0^U(0)} = \frac{P_1\{U = 0\}}{P_0\{U = 0\}} = \frac{P_1\{L(Y) < \eta(\pi_0)\}}{P_0\{L(Y) < \eta(\pi_0)\}} < \eta(\pi_0).
\]

This is because

\[
P_1\{L(Y) \geq \eta(\pi_0)\} = \int_{\{L(y) \geq \eta(\pi_0)\}} p_1(y) dy
\]

\[
\geq \int_{\{L(y) \geq \eta(\pi_0)\}} \eta(\pi_0)p_0(y) dy.
\]

\[
\geq \eta(\pi_0)P_0\{L(Y) \geq \eta(\pi_0)\}.
\]

The inequality on \(L_U(0)\) follows by replacing every \(\geq\) in the above equation by a \(<\). Thus, the Bayes rule for observation \(U\) reduces to

\[
\delta^U_B(y) = \begin{cases} 
1 & \text{if } u = 1 \\
0 & \text{if } u = 0
\end{cases}.
\]

But, \(u = 1\) if \(L(y) \geq \frac{\pi_0}{1-\pi_0}\), and \(u = 0\) if \(L(y) < \frac{\pi_0}{1-\pi_0}\). Thus, this rule has the same Bayes risk as the Bayes rule based on the observation \(Y\).

**Solution B:** Consider the following decision rule based on \(U\), which is equivalent to the Bayes rule based on \(Y\):

\[
\delta^U_B(U) = \begin{cases} 
1 & \text{if } u = 1 \\
0 & \text{if } u = 0
\end{cases}.
\]
We need to show that $\delta^U$ is the Bayes rule based on $U$. The Bayes risk for this test is given by

$$r(\delta^U) = \pi_0 P_0\{U = 1\} + (1 - \pi_0) P_1\{U = 0\}.$$ 

But

$$P_0\{U = 1\} = P_0\{L(Y) \geq \eta(\pi_0)\}$$
$$P_1\{U = 0\} = P_1\{L(Y) < \eta(\pi_0)\}.$$ 

Hence

$$r(\delta^U) = \pi_0 P_0\{L(Y) \geq \eta(\pi_0)\} + (1 - \pi_0) P_1\{L(Y) < \eta(\pi_0)\}.$$ 

But, the right hand side of the above equation is nothing but the minimum possible Bayes risk for this problem with $Y$ as the observation and cannot be made any smaller. Thus $\delta^U$ is the Bayes rule based on $U$. 

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