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**EXAM 1: Solutions**

1. (a) The Bayes solution with uniform costs and equal priors is

$$\delta_{\mathrm{B}}(y) = \begin{cases} 1 & \text{if } L(y) \geq 1 \\ 0 & \text{if } L(y) < 1 \end{cases}$$

Now, $L(0) = 0$, $L(1) = 2/3$, $L(2) = 4/3$, and $L(3) = 2$. Thus, the Bayes rule reduces to

$$\delta_{\mathrm{B}}(y) = \begin{cases} 1 & \text{if } y = 2, 3 \\ 0 & \text{if } y = 0, 1 \end{cases}$$

Note that this is same as the rule $\delta^{(1)}(y)$ of part (c).

(b) The Bayes risk is given by

$$\begin{aligned} r(\delta_{\mathrm{B}}) &= 0.5\ \mathsf{P}_0\{Y \in \{2,3\}\} + 0.5\ \mathsf{P}_1\{Y \in \{0,1\}\} \\ &= 0.5\ (0.5) + 0.5\ (1/6) \\ &= 1/3. \end{aligned}$$

(c) As noted above, $\delta^{(1)} = \delta_{\mathrm{B}}$. Further, $R_0(\delta^{(2)}) = \mathsf{P}_0\{Y = 3\} = 1/4$ and $R_1(\delta^{(2)}) = \mathsf{P}_1\{Y \neq 3\} = 1/2$. Thus,

$$\begin{aligned} r(\pi_0, \delta^{(1)}) &= (1/2)\pi_0 + (1/6)(1 - \pi_0) \\ r(\pi_0, \delta^{(2)}) &= (1/4)\pi_0 + (1/2)(1 - \pi_0). \end{aligned}$$

The risk lines are shown in Fig. 1.

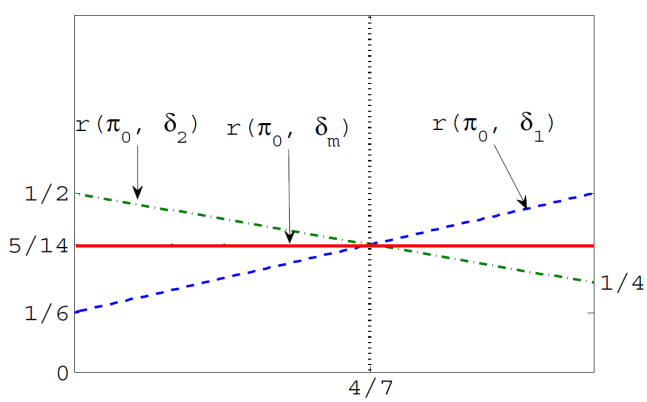


Figure 1: Risk lines for Prob 1 (c).

(d) The risk lines in part (c) have $\pi_0 = 4/7$ as the point of intersection. For this prior, $\frac{\pi_0}{1-\pi_0} = 4/3$. Since, $L(0) = 0$, $L(1) = 2/3$, $L(2) = 4/3$, and $L(3) = 2$, both $\delta^{(1)}$ and $\delta^{(2)}$ are Bayes rules for this prior. The Bayes risk for this prior is equal to $5/14$. An equalizer rule with risk equal to $5/14$ will be a minimax rule. To obtain an equalizer rule, we randomize between the rules $\delta^{(1)}$ and $\delta^{(2)}$.

The randomization parameter can be obtained by solving for $\lambda$ in

$$5/14 = \lambda(1/2) + (1-\lambda)(1/4).$$

This gives $\lambda = 3/7$. Thus the minimax rule is

$$\delta_{\mathrm{m}} = \begin{cases} \delta^{(1)} & \text{w. p. } 3/7 \\ \delta^{(2)} & \text{w. p. } 4/7 \end{cases} = \begin{cases} 1 & \text{if } y = 3 \\ 1 & \text{w. p. } 3/7 \text{ if } y = 2 \\ 0 & \text{if } y = 0, 1 \end{cases}.$$

See Fig. 1.

(e) Due to the monotone nature of $L(y)$ as a function of $y$ and the discrete nature of observations, any Neyman-Pearson rule is of the form

$$\delta_{\mathrm{NP}}(y) = \begin{cases} 1 & \text{if } y > k \\ 1 & \text{w. p. } \gamma \text{ if } y = k \\ 0 & \text{if } y < k \end{cases},$$

for $k = 0, 1, 2, 3$. The threshold $k$ and parameter $\gamma$ is chosen to meet the $\mathrm{P_F}$ constraint of $\alpha$ with equality. The complementary C.D.F. of $Y$ under $H_0$ is given by

$$\mathsf{P}_0\{Y > k\} = 3/4 - 1/4\, k \qquad \text{for } k = 0, 1, 2, 3.$$

Thus, for $\alpha = 1/3$, we have $\mathsf{P}_0\{Y > 2\} < \alpha < \mathsf{P}_0\{Y > 1\}$. This gives threshold $k = 2$ and randomization parameter $\gamma$ as

$$\gamma = \frac{1/3 - 1/4}{1/4} = 1/3.$$

Thus, the NP rule for $\alpha = 1/3$ is equal to

$$\delta_{\mathrm{NP}}(y) = \begin{cases} 1 & \text{if } y = 3 \\ 1 & \text{w. p. } 1/3 \text{ if } y = 2 \\ 0 & \text{if } y = 0, 1 \end{cases}.$$

(f) The NP test of part (e) has the following $\mathrm{P_D}$:

$$\mathrm{P_D}(\delta_{\mathrm{NP}}) = \mathsf{P}_1\{Y = 3\} + 1/3\, \mathsf{P}_1\{Y = 2\} = 11/18.$$

2. (a) We find the NP test for $H_0$ vs $H_1$ for a fixed $\theta > 0$ and check if the rule is independent of $\theta$. There are no point masses (we are dealing with Gaussian distribution), so no randomization is needed. A NP rule is of the form

$$\delta_{\mathrm{NP}}(y) = \begin{cases} 1 & \text{if } \ln L_\theta(y) \geq \eta \\ 0 & \text{otherwise} \end{cases}.$$

The log likelihood can be simplified to

$$\ln L_\theta(y) = y_1\theta + y_2\theta^2 - \theta^2/2 - \theta^4/2.$$

Due to the presence of $\theta^2$ as the coefficient of $y_2$, the dependence on $\theta$ cannot be removed. If $\theta = 1$, the NP rule reduces to

$$\delta_{\mathrm{NP}}(y) = \begin{cases} 1 & \text{if } y_1 + y_2 \geq \eta^{(1)} \\ 0 & \text{otherwise} \end{cases}.$$

If $\theta = 2$, the NP rule reduces to

$$\delta_{\mathrm{NP}}(y) = \begin{cases} 1 & \text{if } y_1 + 2\, y_2 \geq \eta^{(2)} \\ 0 & \text{otherwise} \end{cases}.$$

As a result no UMP solution exists.

(b) The test statistic for the LMP test is given by

$$L_\ell(y) = \frac{\left.\frac{\partial p_\theta(y)}{\partial \theta}\right|_{\theta=0}}{p_0(y)}$$

Now,

$$\frac{\partial p_\theta(y)}{\partial \theta} = p_\theta(y)((y_1 - \theta) + 2\theta(y_2 - \theta^2))$$

Thus,

$$\left.\frac{\partial p_\theta(y)}{\partial \theta}\right|_{\theta=0} = p_0(y)y_1,$$

and

$$L_\ell(y) = \frac{p_0(y)y_1}{p_0(y)} = y_1.$$

Under $H_0$, $Y_1 \sim \mathcal{N}(0,1)$. Thus,

$$\mathrm{P_F} = \mathsf{P}_0\{Y_1 > \eta\} = \alpha$$

imply $\eta = Q^{-1}(\alpha)$, and the $\alpha$-level locally optimum test is given by

$$\delta_{\mathrm{LMP}}(y) = \begin{cases} 1 & \text{if } y_1 \geq Q^{-1}(\alpha) \\ 0 & \text{otherwise} \end{cases}.$$

(c) The $\mathrm{P_D}$ of the LMP test of part (b) for any fixed $\theta > 0$ is given by

$$\mathrm{P_D}(\delta_{\mathrm{LMP}}) = \mathsf{P}_1(Y_1 > Q^{-1}(\alpha)) = Q(|Q^{-1}(\alpha) - \theta|).$$

3. (a) Since $V(\pi_0)$ is a concave function, the partial plot provided is enough to conclude that $V(\pi_0)$ is maximized at $\pi_0 = 0.5$. Moreover the costs are uniform and the risk curve is differentiable at $\pi_0 = 0.5$. Thus, a minimax rule is the Bayes rule for this particular prior $\pi_0 = 0.5$:

$$\delta_{\mathrm{m}}(y) = \begin{cases} 1 & \text{if } L(y) \geq 1 \\ 0 & \text{otherwise} \end{cases}.$$

The minimax risk is the risk of this rule which is 0.3.

(b) The log GLRT statistic is given by

$$\begin{aligned}\ln L_{\mathrm{G}}(y) &= \ln \frac{\sup_\theta p_\theta(y)}{p_0(y)} = \sup_\theta \ln L_\theta(y) \\ &= \max\{\ln L_1(y), \ln L_{-1}(y)\} \\ &= (y - 1/2)\mathbb{1}_{\{y\geq 0\}} + (-y - 1/2)\mathbb{1}_{\{y<0\}} \\ &= |y| - 1/2.\end{aligned}$$

Hence, the GLRT is given by

$$\delta_{\mathrm{GLRT}}(y) = \begin{cases} 1 & \text{if } |y| \leq \eta \\ 0 & \text{if } |y| > \eta \end{cases} \qquad \text{for some } \eta.$$

(c) **Solution A:** The Bayes rule with $U$ as an observation is given by

$$\delta_{\mathrm{B}}^U(y) = \begin{cases} 1 & \text{if } L^U(y) \geq \frac{\pi_0}{1-\pi_0} = \eta(\pi_0) \\ 0 & \text{otherwise} \end{cases},$$

where, $L^U(u)$ is the likelihood ratio of the observation $U = u$ given by $L^U(u) = \frac{p_1^U(u)}{p_0^U(u)}$, with $p_1^U(u)$ and $p_1^U(u)$ as the p.m.fs of $U$ under $H_1$ and $H_0$, respectively. Now note that

$$L^U(1) = \frac{p_1^U(1)}{p_0^U(1)} = \frac{\mathsf{P}_1\{U = 1\}}{\mathsf{P}_0\{U = 1\}} = \frac{\mathsf{P}_1\{L(Y) \geq \eta(\pi_0)\}}{\mathsf{P}_0\{L(Y) \geq \eta(\pi_0)\}} \geq \eta(\pi_0),$$

and

$$L^U(0) = \frac{p_1^U(0)}{p_0^U(0)} = \frac{\mathsf{P}_1(U = 0)}{\mathsf{P}_0(U = 0)} = \frac{\mathsf{P}_1\{L(Y) < \eta(\pi_0)\}}{\mathsf{P}_0\{L(Y) < \eta(\pi_0)\}} < \eta(\pi_0).$$

This is because

$$\begin{aligned}\mathsf{P}_1\{L(Y) \geq \eta(\pi_0)\} &= \int_{\{L(y)\geq\eta(\pi_0)\}} p_1(y)dy \\ &\geq \int_{\{L(y)\geq\eta(\pi_0)\}} \eta(\pi_0)p_0(y)dy. \\ &\geq \eta(\pi_0)\mathsf{P}_0\{L(Y) \geq \eta(\pi_0)\}.\end{aligned}$$

The inequality on $L^U(0)$ follows by replacing every ‘ $\geq'$ in the above equation by a ‘ $<'$. Thus, the Bayes rule for observation $U$ reduces to

$$\delta_{\mathrm{B}}^U(y) = \begin{cases} 1 & \text{if } u = 1 \\ 0 & \text{if } u = 0 \end{cases}.$$

But, $u = 1$ iff $L(y) \geq \frac{\pi_0}{1-\pi_0}$, and $u = 0$ iff $L(y) < \frac{\pi_0}{1-\pi_0}$. Thus, this rule has the same Bayes risk as the Bayes rule based on the observation $Y$.

**Solution B:** Consider the following decision rule based on $U$, which is equivalent to the Bayes rule based on $Y$:

$$\delta^U(u) = \begin{cases} 1 & \text{if } u = 1 \\ 0 & \text{if } u = 0 \end{cases}.$$

We need to show that $\delta^U$ is the Bayes rule based on $U$. The Bayes risk for this test is given by

$$r(\delta^U) = \pi_0 \mathsf{P}_0\{U = 1\} + (1 - \pi_0)\mathsf{P}_1\{U = 0\}.$$

But

$$\begin{aligned} \mathsf{P}_0\{U = 1\} &= \mathsf{P}_0 \left\{L(Y) \geq \eta(\pi_0)\right\} \\ \mathsf{P}_1\{U = 0\} &= \mathsf{P}_1 \left\{L(Y) < \eta(\pi_0)\right\}. \end{aligned}$$

Hence

$$r(\delta^U) = \pi_0 \mathsf{P}_0 \left\{L(Y) \geq \eta(\pi_0)\right\} + (1 - \pi_0)\mathsf{P}_1 \left\{L(Y) < \eta(\pi_0)\right\}.$$

But, the right hand side of the above equation is nothing but the minimum possible Bayes risk for this problem with $Y$ as the observation and cannot be made any smaller. Thus $\delta^U$ is the Bayes rule based on $U$.