University of Illinois **Spring 2017**

ECE 561: Problem Set 6
Information Inequality, MLE, EM

Due: Tuesday, April 18 in class
Reading: Lecture Notes Chapters 12-14

Reminder: Exam 2 will be held on Monday, April 24 from 7-8:30 PM in ECEB 2013. You will be allowed two sheets of notes (8.5"x11", both sides) for the exam. Otherwise the exam is closed book.

1. **[Poisson – II]**
   Suppose $\{Y_k\}_{k=1}^n$ are i.i.d. Poisson random variables with parameter $\theta > 0$, i.e.,
   $$p_\theta(y_k) = \frac{e^{-\theta}\theta^{y_k}}{y_k!} \quad y_k = 0, 1, 2, \ldots$$
   and $p_\theta(\boldsymbol{y}) = \prod_{k=1}^n p_\theta(y_k)$.
   Hint: You may use the fact that under P_θ, both the mean and the variance of Y_k equal θ.
   (a) Find $\hat{\theta}_{\text{ML}}(\boldsymbol{y})$.
   (b) Is $\hat{\theta}_{\text{ML}}$ unbiased? If not, find its bias.
   (c) Find the variance of $\hat{\theta}_{\text{ML}}$.
   (d) Find the CRLB on the variance of unbiased estimators of θ.
   (e) Find an MVUE for θ based on $\boldsymbol{y}$.

2. **[Rayleigh]**
   Suppose $\{Y_k\}_{k=1}^n$ are i.i.d. Rayleigh random variables with parameter $\theta > 0$, i.e.,
   $$p_\theta(y_k) = \frac{2y_k}{\theta} e^{-\frac{y_k^2}{\theta}} \, \mathbb{I}_{\{y_k \geq 0\}}$$
   and $p_\theta(\boldsymbol{y}) = \prod_{k=1}^n p_\theta(y_k)$.
   We wish to estimate $\phi = \theta^2$.
   (a) Find the MLE of ϕ and compute its bias.
   (b) Find the MVUE of ϕ.
   (c) Which of these two estimates has a smaller MSE?
   (d) Compute the CRLB on the variance of unbiased estimators of ϕ.
   (e) Is the MLE asymptotically unbiased as $n \to \infty$?
   (f) Is the MLE asymptotically efficient as $n \to \infty$?

   Hint: You may use the fact that, under P_θ, $X = \sum_{k=1}^n Y_k^2$ has a Gamma pdf given by
   $$p_\theta(x) = \frac{x^{n-1}\, e^{-x/\theta}}{(n-1)!\, \theta^n} \, \mathbb{I}_{\{x \geq 0\}}$$

3. **[Estimating a line in noise]**
   Suppose
   $$Y_k = \theta_1 + \theta_2 k + Z_k, \;\; k = 1, 2, \ldots, n$$
   where $\{Z_k\}$ are i.i.d. $\mathcal{N}(0, 1)$ random variables.

(a) Find the Fisher information matrix for the vector parameter $\boldsymbol{\theta} = [\theta_1\ \theta_2]$ and its inverse. You may leave your answer in terms of the sums:

$$s_1(n) = \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \qquad s_2(n) = \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$$

(b) Now suppose $n = 2$. Find the ML estimate of $\boldsymbol{\theta}$.

(c) Show that the ML estimate of part (c) is unbiased.

(d) Show that the ML estimate of part (c) is efficient, i.e., that the covariance of the ML estimate equals the inverse of the Fisher information matrix.

4. **[E-M Algorithm for estimating the parameter of an exponential distribution]**
Suppose that given a parameter $\theta > 0$, the random variable X is exponential with parameter θ, i.e.,

$$p_\theta(x) = \theta e^{-\theta x}\ \mathbb{I}_{\{x \geq 0\}}$$

The random variable X is not observed directly, rather a noisy version Y is observed, with

$$Y = X + W$$

where W is exponential with parameter β, and independent of X.

Our goal is to estimate θ from y.

(a) Find the likelihood function $p_\theta(y)$ and write down the corresponding likelihood equation.

(b) Can you find $\hat{\theta}_{\text{ML}}(y)$ in closed-form? If your answer is yes, provide the solution.

(c) Now consider computing $\hat{\theta}_{\text{ML}}(y)$ iteratively using the E-M algorithm based on $Z = (Y, X)$. Compute the E-Step of the algorithm, i.e., find $Q(\theta|\hat{\theta}^{(k)})$.

(d) Compute the M-step in closed form, i.e., find $\hat{\theta}^{(k+1)}$ in terms of $\hat{\theta}^{(k)}$ and y.

5. **[E-M Algorithm for interference cancellation]**
Consider the observation model:

$$\boldsymbol{Y}_1 = \boldsymbol{X}_1 + \theta \boldsymbol{X}_2 + \boldsymbol{W}_1$$

where $\boldsymbol{X}_1 \sim \mathcal{N}(\underline{0}, \mathsf{C}_1)$ is the desired signal, $\boldsymbol{X}_2 \sim \mathcal{N}(\mathbf{0}, \mathsf{C}_2)$ is additive interference, and $\boldsymbol{W}_1 \sim \mathcal{N}(\mathbf{0}, \sigma^2 I)$ is white noise. We wish to cancel $\boldsymbol{X}_2$ by placing a sensor close to the interferer that produces observation:

$$\boldsymbol{Y}_2 = \boldsymbol{X}_2 + \boldsymbol{W}_2$$

where $\boldsymbol{W}_2 \sim \mathcal{N}(\underline{0}, \sigma^2 I)$ is white noise.

Assume that $\boldsymbol{X}_1$, $\boldsymbol{X}_2$, $\boldsymbol{W}_1$ and $\boldsymbol{W}_2$ are independent, and that C_1, C_2 and σ^2 are known.

To cancel $\boldsymbol{X}_2$ in $\boldsymbol{Y}_1$, we need to estimate θ. Develop an E-M algorithm to estimate θ from $\boldsymbol{Y}_1$, $\boldsymbol{Y}_2$.