University of Illinois **Spring 2017**

ECE 561: Problem Set 5
Bayesian Estimation, MVUE

Due: Tuesday, April 4 in class
Reading: Lecture Notes Chapters 10,11

1. **[MAP and MMSE]**
Suppose Λ is a random parameter with prior given by the Gamma density

$$\pi(\lambda) = \frac{e^{-\lambda}\ \lambda^{\alpha-1}}{\Gamma(\alpha)}\ \mathbb{I}_{\{\lambda \geq 0\}}$$

where α is a known positive real number, and Γ is the Gamma function defined by the integral

$$\Gamma(x) = \int_0^\infty t^{x-1}\ e^{-t}\ dt\ , \quad \text{for } x > 0\ .$$

Our observation Y is Poisson with rate Λ, i.e.,

$$p_\lambda(y) = \mathrm{P}(\{Y = y\} \mid \{\Lambda = \lambda\}) = \frac{\lambda^y\ e^{-\lambda}}{y!}\ , \quad y = 0, 1, 2, \ldots\ .$$

(a) Find the MAP estimate of λ given y.

(b) Now suppose we wish to estimate a parameter Θ that is related to Λ as

$$\Theta = e^{-\Lambda}$$

Find the MMSE estimate of θ given y

2. **[MAP/MMSE/MMAE]**
Suppose Θ is uniformly distributed on $[0, 1]$ and that we observe $Y = \Theta + Z$, where Z is a random variable, independent of Θ, with exponential pdf

$$p_Z(z) = e^{-z}\mathbb{I}_{\{z \geq 0\}}.$$

Find $\hat{\theta}_{\mathrm{MMSE}}(y)$, $\hat{\theta}_{\mathrm{MMAE}}(y)$, and $\hat{\theta}_{\mathrm{MAP}}(y)$.

3. **[MAP for i.i.d. observations]**
Let $Y_1, ..., Y_n$ be iid observations drawn from a uniform distribution over $[1, \theta]$, i.e.,

$$p_\theta(y_k) = \frac{1}{\theta - 1}\mathbb{I}_{\{y_k \in [1,\theta]\}}.$$

The prior distribution of θ is given by a uniform distribution over $[1, 10]$, i.e.,

$$\pi(\theta) = \frac{1}{9}\mathbb{I}_{\{\theta \in [1,10]\}}.$$

(a) Find $\hat{\theta}_{\mathrm{MAP}}(\boldsymbol{y})$.

(b) Show that $\hat{\theta}_{\mathrm{MAP}}(\boldsymbol{y})$ in distribution to the true value of θ as $n \to \infty$, i.e., show that

$$\lim_{n\to\infty} \mathrm{P}\{\hat{\theta}_{\mathrm{MAP}}(\boldsymbol{Y}) \leq t\} = \mathrm{P}\{\Theta \leq t\} \ \text{ for all } t \in \Re.$$

4. **[Sufficient Statistics]**

   (a) Consider data $\boldsymbol{Y} = [Y_1, Y_2, \cdots, Y_n]$, and the reordered data $T(\boldsymbol{Y}) = [Y_{(1)}, Y_{(2)}, \cdots, Y_{(n)}]$, where $Y_{(1)} \geq Y_{(2)} \geq \cdots \geq Y_{(n)}$. Clearly T is a many-to-one map. Show that if $Y_1, Y_2, \cdots, Y_n$ are iid with marginals $p_\theta(y)$, then $T(\boldsymbol{Y})$ is a sufficient statistic for θ.

   (b) Suppose that given a parameter $\theta > 1$, $Y_1, Y_2, \ldots, Y_n$ are i.i.d. observations each with pdf

$$p_\theta(y) = (\theta - 1) y^{-\theta} \mathbb{I}_{\{y \geq 1\}}$$

   That is,

$$p_\theta(\boldsymbol{y}) = \prod_{k=1}^{n} (\theta - 1) y_k^{-\theta} \mathbb{I}_{\{y_k \geq 1\}}$$

   Find a complete sufficient statistic for $\{p_\theta, \theta > 1\}$

5. **[Uniform]**
   Suppose $\theta > 0$ is a parameter of interest and that given θ, $Y_1, \ldots, Y_n$ are i.i.d. observations with marginal densities

$$p_\theta(y) = \begin{cases} \frac{1}{\theta} \text{ if } y \in [0, \theta] \\ 0 \qquad\qquad \text{otherwise} \end{cases}.$$

   (a) Prove that $T(\boldsymbol{y}) = \max\{y_1, \ldots, y_n\}$ is a sufficient statistic for θ.

   (b) Prove that $T(\boldsymbol{y})$ is also *complete*. Note that you cannot apply the Completeness Theorem for Exponential Families.

$$\text{Hint: Use the fact that } \int_0^\theta f(t) \frac{t^{n-1}}{\theta^n} dt \;=\; 0,\ \forall\, \theta > 0 \quad \Rightarrow \quad f(t) = 0 \text{ on } [0, \infty)$$

   (c) Find an MVUE of θ based on $\boldsymbol{y}$.

6. **[Poisson – I]**
   Suppose $\{Y_k\}_{k=1}^n$ (with $n \geq 2$) are i.i.d. Poisson random variables with parameter (mean) $\theta > 0$, i.e.,

$$p_\theta(y_k) = \frac{e^{-\theta} \theta^{y_k}}{y_k!} \quad y_k = 0, 1, 2, \ldots$$

   and $p_\theta(\boldsymbol{y}) = \prod_{k=1}^n p_\theta(y_k)$.
   We wish to estimate the parameter λ that is given by

$$\lambda = e^{-\theta}$$

   (a) Show that $T(\boldsymbol{y}) = \sum_{k=1}^n y_k$ is a complete sufficient statistic for this estimation problem.

   (b) Show that

$$\hat{\lambda}_1(\boldsymbol{y}) = \frac{1}{n} \sum_{k=1}^{n} \mathbb{I}_{\{y_k = 0\}}$$

   is an unbiased estimator of λ.

   (c) Show that

$$\hat{\lambda}_2(\boldsymbol{y}) = \left[\frac{n-1}{n}\right]^{T(\boldsymbol{y})}$$

   is an MVUE of λ.
   Hint: You may use the fact that $T(\boldsymbol{y})$ is Poisson with parameter $n\theta$.