University of Illinois **Spring 2017**

ECE 561: Problem Set 4
Chernoff Bound, Large Deviations, Error Exponents

Due: Tuesday, March 14 in class
Reading: Lecture Notes Chapters 6 and 8.

1. **[Chernoff and Bhattacharya Bounds]**
   Consider the binary hypothesis testing problem with

   $$p_0(y) = \frac{1}{2}e^{-|y|} \quad \text{and} \quad p_1(y) = e^{-2|y|}$$

   Assume equal priors.

   (a) Find $\mathrm{P_e}$ for Bayes rule.

   (b) Find the Bhattacharya bound on $\mathrm{P_e}$.

   (c) Find the Chernoff bound on $\mathrm{P_e}$.

2. **[Chernoff Information and K-L Divergence]**
   Recall that the Chernoff information for p_0 and p_1 is given by

   $$C(p_0, p_1) \triangleq \max_{u \in [0,1]} -\ln \int_{\mathcal{Y}} p_0^{1-u}(y) p_1^u(y) d\mu(y)$$

   Now define the "geometric mixture" of p_0 and p_1 by:

   $$p_u(y) \triangleq \frac{p_0^{1-u}(y) p_1^u(y)}{\int_{\mathcal{Y}} p_0^{1-u}(y) p_1^u(y) d\mu(y)}$$

   Show that the optimizing value of u in definition of $C(p_0, p_1)$ satisfies the equation:

   $$C(p_0, p_1) = D(p_{u^*} \| p_0) = D(p_{u^*} \| p_1)$$

   Hint: You may want to use the fact that $p_u(y)$ can be written as:

   $$p_u(y) = \frac{p_0(y)\ L(y)^u}{\mathbb{E}_0[L(Y)^u]}$$

3. **[Slight Generalization of Cramer's Theorem]**
   Let $X_1, X_2, \ldots,$ be i.i.d. with mean $\mathbb{E}[X]$, and let $S_n = \sum_{k=1}^n X_k$. Then, for $a > \mathbb{E}[X]$, we know from class that

   $$\mathrm{P}\{S_n \geq na\} \leq \exp(-n\Lambda_X(a))$$

   and that, given $\epsilon > 0$, there exists n_ϵ such that

   $$\mathrm{P}\{S_n \geq na\} \geq \exp(-n(\Lambda_X(a) + \epsilon)) \quad \text{for all } n > n_\epsilon.$$

   These two bounds establish Cramér's Theorem. Now use these bounds and the continuity of Λ_X to show the following generalization of this result. Suppose $a_n \to a$ as $n \to \infty$ and $a > \mathbb{E}[X]$, then

   $$\lim_{n\to\infty} \frac{1}{n} \ln \mathrm{P}\{S_n \geq na_n\} = -\Lambda_X(a)$$

4. **[Gaussian – I]**
   Consider the detection problem where the observations, $\{Y_k, k \geq 1\}$ are i.i.d. $\mathcal{N}(\mu_j, \sigma^2)$ random variables under H_j, $j = 0, 1$.

(a) Evaluate the cumulant generating function $\kappa_0(u)$.

(b) Use $\kappa_0(u)$ to find $D(p_0 \| p_1)$ and $D(p_1 \| p_0)$.

(c) Find the rate functions $\Lambda_0(\tau)$ and $\Lambda_1(\tau)$.

(d) Find the Chernoff information $C(p_0, p_1)$.

(e) Solve the Hoeffding problem when the constraint on the false alarm exponent is equal to γ, for $\gamma \in (0, D(p_1 \parallel p_0))$.

5. **[Gaussian - II]**
Repeat parts (a)-(d) of Problem 4 for the case where the observations, $\{Y_k, k \geq 1\}$ are i.i.d. $\mathcal{N}(0, \sigma_j^2)$ random variables under H_j, $j = 0, 1$.