ECE 561: Problem Set 3
Signal Detection, Sequential and Quickest Change Detection

Due: Tuesday, Feb 28 in class
Reading: Lecture Notes Chapters 5,7; Poor, Chapter 2 and 3; Levy, Chapter 2 and 3.

REMINDER: Exam 1 will be held in (**ECEB 2013** on **Wednesday, March 8** from 7:00 PM - 8:30 PM. You will be tested on all material covered in class until Thursday, February 23. You will be allowed one sheet of handwritten notes (8.5×11 inches, both sides).

1. [**Signal Detection in Correlated Gaussian Noise - I**]
Consider the detection problem with

$$\begin{cases} H_0: & \boldsymbol{Y} = \begin{bmatrix} -a \\ 0 \end{bmatrix} + \boldsymbol{Z} \\ H_1: & \boldsymbol{Y} = \begin{bmatrix} a \\ 0 \end{bmatrix} + \boldsymbol{Z} \end{cases}$$

where $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \mathsf{C}_{\boldsymbol{Z}})$ with

$$\mathsf{C}_{\boldsymbol{Z}} = \begin{bmatrix} 1 & \rho \\ \rho & 1+\rho^2 \end{bmatrix}.$$

Assume that $a > 0$ and $\rho \in (0,1)$.

   (a) For equal priors show that the minimum-probability-of-error detector is given by

$$\delta_{\text{B}}(\boldsymbol{y}) = \begin{cases} 1 & \text{if } y_1 - by_2 \geq \tau \\ 0 & \text{if } y_1 - by_2 < \tau \end{cases}$$

   where $b = \rho/(1+\rho^2)$ and $\tau = 0$.

   (b) Determine the minimum probability of error.

   (c) Consider the test of part (a) in the limit as $\rho \to 0$. Explain why the dependence on y_2 goes away in this limit.

   (d) Now suppose the observations $\boldsymbol{Y} \sim \mathcal{N}([a\ 0]^\top, \mathsf{C}_{\boldsymbol{Z}})$, with $\rho = 1$ but a being an unknown parameter, and we wish to test between the hypotheses:

$$\begin{cases} H_0 & : \ 0 < a \leq 1 \\ H_1 & : \ a > 1 \end{cases}$$

   Show that a UMP test exists for this problem, and find the UMP test of level $\alpha \in (0,1)$.

2. [**Signal Detection in Correlated Gaussian Noise - II**]
Consider the hypothesis testing problem with n-dimensional observations:

$$\begin{cases} H_0: & \boldsymbol{Y} = \boldsymbol{Z} \\ H_1: & \boldsymbol{Y} = \boldsymbol{s} + \boldsymbol{Z} \end{cases}$$

where the components of $\boldsymbol{Z}$ are zero mean correlated Gaussian random variables with

$$\mathbb{E}[Z_k Z_\ell] = \sigma^2\ \rho^{|k-\ell|}, \quad \text{for all } 1 \leq k, \ell \leq n$$

where $|\rho| < 1$.

(a) Show that the N-P test for this problem has the form:

$$\delta_\tau(\boldsymbol{y}) = \begin{cases} 1 & \text{if } \sum_{k=1}^n b_k x_k \geq \tau \\ 0 & \text{if } \sum_{k=1}^n b_k x_k < \tau \end{cases}$$

where $b_1 = s_1/\sigma$, $x_1 = y_1/\sigma$, and

$$b_k = \frac{s_k - \rho s_{k-1}}{\sigma\sqrt{1-\rho^2}}, \; x_k = \frac{y_k - \rho y_{k-1}}{\sigma\sqrt{1-\rho^2}}, \quad k = 2, \ldots, n.$$

Hint: Note that $\mathsf{C}_{\boldsymbol{Z}}^{-1} = \mathsf{A}/(\sigma^2(1-\rho^2))$, where A is a tridiagonal matrix with main diagonal $(1 \;\; 1+\rho^2 \;\; 1+\rho^2 \;\; \ldots \;\; 1+\rho^2 \;\; 1)$ and superdiagonal and subdiagonal entries all being $-\rho$.

(b) Find an α-level NP test.

(c) Find the ROC for the above detector. (You don't need to plot it, just give the expression for $\mathrm{P_D}$ as a function of level α.)

3. **[LMP detection of a Gaussian Signal in White Gaussian Noise.]**
Consider the detection problem:

$$\begin{cases} H_0: \; \boldsymbol{Y} = \boldsymbol{Z} \\ H_1: \; \boldsymbol{Y} = \sqrt{\theta}\boldsymbol{S} + \boldsymbol{Z} \end{cases}$$

where $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \mathsf{I}_n)$, $\boldsymbol{S} \sim \mathcal{N}(\boldsymbol{0}, \mathsf{C}_{\boldsymbol{S}})$, and $\theta > 0$ is an unknown parameter.

(a) Show that locally most powerful (LMP) test as $\theta \downarrow 0$ uses the quadratic test statistic:

$$T_{\mathrm{LMP}}(\boldsymbol{y}) = \boldsymbol{y}^\top \mathsf{C}_{\boldsymbol{S}} \boldsymbol{y}$$

Hint: Spectral factorization of $\mathsf{C}_{\boldsymbol{S}}$ may be useful here.

(b) Show that the quadratic statistic that maximizes the deflection is independent of θ and equals $T_{\mathrm{LMP}}(\boldsymbol{y})$.

4. **[Baseband Detection of a Sinusoid with Drifting Phase]**
Consider the detection problem:

$$\begin{cases} H_0: \; \boldsymbol{Y} = \boldsymbol{Z} \\ H_1: \; \boldsymbol{Y} = \boldsymbol{S} + \boldsymbol{Z} \end{cases}$$

where

- The signal $S_k = \sqrt{2}\cos(\Theta_k + \phi)$ is a sinusoid with randomly varying phase.
- The noise is white Gaussian, i.e., $\boldsymbol{Z} \sim \mathcal{N}(\boldsymbol{0}, \mathsf{I}_n)$.
- $\Theta_k = \sqrt{-2\ln\beta} \displaystyle\sum_{i=1}^{k} W_i$, where $0 < \beta < 1$ and $\{W_i\}$ are i.i.d. $\mathcal{N}(0,1)$ independent of $\boldsymbol{Z}$. That is, the phase Θ_k drifts as a random walk.
- ϕ is uniform on $[0, 2\pi]$, and is independent of $\boldsymbol{Z}$ and $\boldsymbol{W}$.

(a) Prove that $\mathbb{E}[\boldsymbol{S}] = 0$, and that

$$\mathbb{E}[S_k S_\ell] = \beta^{|k-\ell|}, \quad 1 \leq k, \ell \leq n$$

Hint: You may want to use the fact that the characteristic function of a $\mathcal{N}(0, \sigma^2)$ random variable X is given by $\mathbb{E}[e^{\imath u X}] = e^{-\sigma^2 u^2/2}$.

(b) Note that the signal is not Gaussian and so it is difficult to compute the likelihood ratio test. Give an expression for the quadratic test statistic that maximizes the deflection and show that this maximum deflection is given by:

$$D_{\max} \triangleq f(\beta, n) = \frac{n}{2} + \frac{n\beta^2}{1-\beta^2} - \frac{\beta^2(1-\beta^{2n})}{(1-\beta^2)^2}$$

Hint: Use the fact that for any function g defined for non-negative integers

$$\sum_{k=1}^{n}\sum_{\ell=1}^{n} g(|k-\ell|) = ng(0) + 2\sum_{i=1}^{n-1}(n-i)g(i)$$

(c) Now consider the Standard Noncoherent Detector (SND) that uses the test statistic

$$T_{\text{SND}}(\boldsymbol{y}) = \left(\sum_{k=1}^{n} y_k\right)^2$$

Show that the deflection for this statistic is given by

$$D(T_{\text{SND}}) = \frac{2\left[f(\sqrt{\beta}, n)\right]^2}{n^2}$$

where $f(\cdot)$ is defined in part (b).

(d) Compare $D_{\max}$ and $D(T_{\text{SND}})$ for $n = 100$ and $\beta = 0.9, 0.5$ and 0.1. Explain why the performance of the SND degrades rapidly as β goes to zero.

5. **[Sequential Hypothesis Testing]**
Consider the problem of sequentially testing between the distributions

$$p_1(y_k) = \begin{cases} \frac{3}{4} & \text{if } y_k = 1 \\ \frac{1}{4} & \text{if } y_k = 0 \end{cases}, \quad p_0(y_k) = \begin{cases} \frac{1}{4} & \text{if } y_k = 1 \\ \frac{3}{4} & \text{if } y_k = 0 \end{cases}$$

using an SPRT with thresholds $a < 0 < b$ on the log-likelihood ratio.

(a) Let $N_1(n)$ denote the number of observations that are equal to 1 among the first n observaions. Show that the SPRT can be written in terms of $N_1(n)$.

(b) Suppose $a = -10\ln 3$ and $b = 10\ln 3$. Show that Wald's approximations for the error probabilities and expected stopping times are exact in this case.

(c) For the above choice of thresholds a and b, find the error probabilities $\mathrm{P_F}$ and $\mathrm{P_M}$.

(d) For the above choice of thresholds a and b, find $\mathbb{E}_0[N]$ and $\mathbb{E}_1[N]$.

6. **[SPRT vs FSS Simulation]**
Consider the detection problem:

$$\begin{cases} H_0 : Y_1, Y_2, \ldots, \text{ are i.i.d. } \mathcal{N}(-0.25, 1) \\ H_1 : Y_1, Y_2, \ldots, \text{ are i.i.d. } \mathcal{N}(0.25, 1) \end{cases}$$

Your goal is to design tests that have error probabilies $\alpha = 0.05$ and $\beta = 0.05$.

(a) Design a fixed sample size test that has the desired error probabilities. That is, find τ and n_{FSS} such that the likelihood ratio test

$$\delta(y_1, \ldots, y_{n_{\text{FSS}}}) = \begin{cases} 1 & \text{if } \sum_{k=1}^{n_{\text{FSS}}} y_k \geq \tau \\ 0 & \text{if } \sum_{k=1}^{n_{\text{FSS}}} y_k < \tau \end{cases}$$

has the desired error probabilities.

(b) Design an SPRT for the same α and β using Wald's approximations. Also compute the corresponding Wald approximations for $\mathbb{E}_0[N]$ and $\mathbb{E}_1[N]$.

(c) Simulate the performance of the SPRT using at least 1000 trials and compare your answers for the error probabilities and expected stopping times with part (b).

(d) Truncate the SPRT to a maximum of n_{FSS} observations in the simulations of part (c) and find the resulting error probabilities and expected stopping times for this truncated test.

7. **[CuSum Recursion]**
Prove that the CuSum statistics W_n and C_n can be computed iteratively as given in the notes, i.e.,

$$W_n = (W_{n-1} + \ln(Y_n))^+, \quad W_0 = 0.$$

and

$$C_n = (C_{n-1})^+ + \ln(Y_n), \quad C_0 = 0.$$

Also use these recursions to conclude that W_n and C_n will cross a positive threshold b at the same time (sample-path wise).

8. **[Shiryaev and SR Recursions]**

(a) Show that the Shiryaev statistic G_n satisfies the recursion

$$G_n = \Phi(Y_n, G_{n-1}),$$

where

$$\Phi(Y_n, G_{n-1}) = \frac{\tilde{G}_{n-1} L(Y_n)}{\tilde{G}_{n-1} L(Y_n) + (1 - \tilde{G}_{n-1})},$$

$\tilde{G}_{n-1} = G_{n-1} + (1 - G_{n-1})\rho$, and $G_0 = 0$.

(b) Show that the SR statistic can be computed iteratively as:

$$T_n = (1 + T_{n-1}) L(Y_n), \quad T_0 = 0.$$