University of Illinois **Spring 2017**

ECE 561: Problem Set 2
Neyman-Pearson HT, M-ary HT, Composite HT

Due: Tuesday, February 14 in class
Reading: Lecture Notes Chapters 2, 3, 4; Poor, Chapter 2; Levy, Chapter 2.

1. **[Neyman-Pearson Hypothesis Testing – Continuous]**
   Consider the binary detection problem with
   $$p_0(y) = \frac{1}{2}e^{-|y|} \quad \text{and} \quad p_1(y) = \frac{1}{2}e^{-|y-2|}, \quad y \in \mathbb{R}$$
   This is a case where the likelihood ratio has point masses even though p_0 and p_1 have no point masses.
   (a) Show that the Neyman-Pearson solution, which is a randomized LRT, is equivalent to a deterministic *threshold* test on the observation y.
   (b) Find the ROC for the test in part (a).

2. **[Neyman-Pearson Hypothesis Testing – Discrete]**
   Consider the detection problem for which $\mathcal{Y} = \{0, 1, 2, \ldots\}$ and the pmf's of the observations under the two hypotheses are:
   $$p_0(y) = (1 - \beta_0)\beta_0^y, \; y = 0, 1, 2, \ldots$$
   and
   $$p_1(y) = (1 - \beta_1)\beta_1^y, \; y = 0, 1, 2, \ldots.$$
   Assume that $0 < \beta_0 < \beta_1 < 1$.
   (a) Find a Bayes rule for uniform costs and equal priors.
   (b) Find the Neyman-Pearson rule with false-alarm probability $\alpha \in (0, 1)$.
   (c) Find the corresponding probability of detection for the N-P test as a function of α.

3. **[Properties of the ROC Curve]**
   Consider the detection problem where $L(y)$ has *no point masses* under either hypothesis. Let δ_η denote the likelihood ratio test:
   $$\delta_\eta(y) = \begin{cases} 1 & \text{if } L(y) \geq \eta \\ 0 & \text{if } L(y) < \eta \end{cases}.$$
   A plot of $\mathrm{P_D}(\delta_\eta)$ versus $\mathrm{P_F}(\delta_\eta)$ for various values of η is called the receiver operating characteristic (ROC). This plot is a concave function with the point $(0, 0)$ corresponding to $\eta = \infty$, and the point $(1, 1)$ corresponding to $\eta = 0$. Prove the following properties of ROC's:
   (a) $\mathrm{P_D}(\delta_\eta) \geq \mathrm{P_F}(\delta_\eta)$ for all η.
   Hint: One way to solve this problem is to consider the cases $\eta \leq 1$ and $\eta > 1$ separately and use the following change of measure argument: For any event A,
   $$P_1(A) = \int_A p_1(y) d\mu(y) = \int_A L(y) p_0(y) d\mu(y)$$
   (b) The slope of the ROC at a particular point is equal to the value of the threshold η required to achieve the $\mathrm{P_D}$ and $\mathrm{P_F}$ at that point, i.e.,
   $$\frac{d\mathrm{P_D}}{d\mathrm{P_F}} = \eta.$$

Hint: Start with

$$\frac{d\mathrm{P_D}}{d\mathrm{P_F}} = \frac{\frac{d\mathrm{P_D}}{d\eta}}{\frac{d\mathrm{P_F}}{d\eta}},$$

and then express $\mathrm{P_D}$ and $\mathrm{P_F}$ in terms of the pdf of $L(Y)$ under H_0 using a change of measure argument similar to that in part (a) above.

4. **[Ternary Hypothesis Testing]**
Consider the ternary hypothesis testing problem where p_0, p_1 and p_2 are uniform pdf's over the intervals $[0,1]$, $[0,\frac{1}{2})$, and $[\frac{1}{2},1]$, respectively. Assume uniform costs.

   Find the Bayes rule and the minimum Bayes risk assuming a prior $\underline{\pi} = [\pi_0, \frac{1-\pi_0}{2}, \frac{1-\pi_0}{2}]$, for all values of $\pi_0 \in [0,1]$.

5. **[Bayesian Composite Hypothesis Testing]**
Consider the composite binary hypothesis testing problem in which

$$p_\theta(y) = \begin{cases} \theta e^{-\theta y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

   and

$$\pi(\theta) = \begin{cases} 1 & \text{if } 0 \leq \theta \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

   Find a Bayes rule and corresponding minimum Bayes risk for the hypotheses

$$\begin{aligned} H_0 &: \mathcal{X}_0 = [0, 0.5) \\ H_1 &: \mathcal{X}_1 = [0.5, 1] \end{aligned}$$

   Assume uniform costs.

6. **[Monotone Likelihood Ratio Theorem with Both Hypotheses being Composite]**
Consider the composite detection problem:

$$\begin{aligned} H_0 &: \theta \in \mathcal{X}_0 \\ H_1 &: \theta \in \mathcal{X}_1 \end{aligned}$$

   Now suppose that for each fixed $\theta_0 \in \mathcal{X}_0$ and each fixed $\theta_1 \in \mathcal{X}_1$, we have

$$\frac{p_{\theta_1}(y)}{p_{\theta_0}(y)} = F_{\theta_0,\theta_1}(g(y))$$

   where the function g does not depend on θ_1 or θ_0, and the function F_{θ_0,θ_1} is *strictly increasing* in its argument.

   Show that for any level $\alpha \in (0,1)$, a UMP test between H_1 and H_0 exists.

7. **[UMP/LMP testing with Laplacian Observations]**
Consider the composite binary detection problem in which

$$p_\theta(y) = \frac{1}{2} e^{-|y-\theta|}, \; y \in \mathbb{R}.$$

   and

$$\begin{aligned} H_0 &: \theta = 0 \\ H_1 &: \theta > 0 \end{aligned}$$

(a) Show that a UMP test exists for this problem, even though the Monotone Likelihood Ratio Theorem discussed in class does not apply. Find an α-level UMP test, and find $\mathrm{P_D}$ as a function of α for this test.

Hint: You may need to write the NP LRT as a deterministic threshold test on y (see Problem 1) in order to establish the existence of a UMP.

(b) Find a locally most powerful α-level test, and find $\mathrm{P_D}$ as a function of α for this test.

(c) Plot the ROCs for the tests in parts (a) and (b) on the same figure and compare them.

8. **[UMP versus GLRT]**
Consider the composite binary detection problem in which

$$p_\theta(y) = \begin{cases} \theta e^{-\theta y} & \text{if } y \geq 0 \\ 0 & \text{if } y < 0 \end{cases}$$

(a) For $\alpha \in (0,1)$, show that a UMP test of level α exists for testing the hypotheses

$$\begin{aligned} H_0 &: \mathcal{X}_0 = [1,2] \\ H_1 &: \mathcal{X}_1 = (2,\infty) \end{aligned}$$

Find this UMP test as a function of α.

(b) Find an α-level generalized likelihood ratio test for testing between H_0 and H_1.