University of Illinois **Spring 2017**

ECE 561: Problem Set 1
Statistical Decision-Making Framework, Binary Bayesian and Minimax Hypothesis Testing

Due: Tuesday, January 31 in class
Reading: Lecture Notes Chapters 1 and 2; Poor, Chapter 2.

1. **[Statistical Decision Making]**
Consider the binary statistical decision theory problem for which $\mathcal{X} = \mathcal{A} = \{0, 1\}$. Suppose the cost function is given by

$$C(a, x) = \begin{cases} 0 & \text{if } a = x \\ 1 & \text{if } x = 0, a = 1 \\ 10 & \text{if } x = 1, a = 0 \end{cases}$$

The observation Y takes values in the set $\mathcal{Y} = \{1, 2, 3\}$ and the conditional p.m.f.'s of Y are:

$$p_0(1) = 0.4, p_0(2) = 0.6, p_0(3) = 0, \qquad p_1(1) = p_1(2) = 0.25,\ p_1(3) = 0.5$$

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   (a) Is there a best decision rule based on conditional risks?

   (b) Find Bayes (for equal priors) and minimax rules within the set of deterministic decision rules.

   (c) Now consider the set of randomized decision rules. Find a Bayes rule (for equal priors). Also find a randomized rule whose maximum risk is smaller than that of the minimax rule of part (b).

2. **[Optimal Rule Based on Conditional Risks]**
For the binary hypothesis testing problem, with $C_{0,0} < C_{1,0}$ and $C_{1,1} < C_{0,1}$, show there is no "best" rule based on conditional risks, except in the trivial case case where $p_0(y)$ and $p_1(y)$ have disjoint supports.

3. **[Health Insurance]**
Find the minimax decision rule for the health insurance problem given in Example 1.3 on page 13 of the notes.

4. **[Binary Communication with Erasures]**
Let $\mathcal{X} = \{0, 1\}$, and $\mathcal{A} = \{0, 1, e\}$. This would correspond to binary communication with erasures. Now suppose

$$p_j(y) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left[-\frac{(y - (-1)^{j+1})^2}{2\sigma^2}\right], \quad j = 0, 1, \ -\infty < y < \infty.$$

That is, Y has distribution $\mathcal{N}(-1, \sigma^2)$ when the state is 0, and Y has distribution $\mathcal{N}(1, \sigma^2)$ when the state is 1. Assume a cost structure

$$C_{i,j} = \begin{cases} 0 & \text{if } i = 0, j = 0 \text{ or } i = 1, j = 1 \\ 1 & \text{if } i = 1, j = 0 \text{ or } i = 0, j = 1 \\ c & \text{if } i = e \end{cases}$$

Furthermore, assume that the two states are equally likely.

   (a) First assume that $c < 0.5$. Show that the Bayes rule for this problem has the form:

$$\delta_B(y) = \begin{cases} 0 & y \le -t \\ e & -t < y < t \\ 1 & y \ge t \end{cases}$$

   Also give an expression for t in terms of the parameters of the problem.

(b) Now find $\delta_B(y)$ when $c \geq 0.5$.

5. **[Minimum Bayes Risk Curve and Minimax Rule]**
For problem 1 above, find the minimum Bayes risk function $V(\pi_0)$, and then find a minimax rule in the set of randomized decision rules using $V(\pi_0)$.

6. **[Binary Hypothesis Testing]**
Consider the binary detection problem with

$$p_1(y) = \begin{cases} \frac{3y^2}{2} & \text{if } y \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}, \quad \text{and} \quad p_0(y) = \begin{cases} \frac{1}{2} & \text{if } y \in [-1, 1] \\ 0 & \text{otherwise} \end{cases}$$

(a) Find a Bayes rule for uniform costs and equal priors and the corresponding minimum Bayes risk.

(b) Find a minimax rule for uniform costs, and the corresponding minimax risk.

7. **[Binary Hypothesis Testing with Nonuniform Costs]**
Consider the binary detection problem with

$$p_0(y) = \frac{1}{2}e^{-|y|} \quad \text{and} \quad p_1(y) = e^{-2|y|}, \quad y \in \mathbb{R}$$

(a) Find the Bayes rule for equal priors and a cost structure of the form $C_{00} = C_{11} = 0$, $C_{10} = 1$ and $C_{01} = 4$.

(b) Find the Bayes risk for the Bayes rule of part (a). (Note that the costs are not uniform.)