**ECE 559BH Distributed Network Algorithms** **SPRING 2006**
**PROBLEM SET 3** **Due 3/7/06**

**Randomized online algorithms, martingale stopping time results, and applications**

**Reading:** See website.

**Problems to be handed in:**

**1. Lower bounding performance of random on-line algorithms by Yao's method (Xiolan Zhang)**
(a) Consider an online problem such as the paging problem. Let the following two assumptions hold:

> **Assumption A1:** A random sequence of arrivals can be generated according to some probability distribution $P$, and the following limits exist and are finite, where $f_A$ denotes the performance of an online deterministic algorithm $A$, and $f_O$ denotes the performance of an optimal offline algorithm:
>
> $$\lim_{N\to\infty} \frac{1}{N} E_P[f_A(\Phi_N)] \quad \text{and} \quad \lim_{N\to\infty} \frac{1}{N} E_P[f_O(\Phi_N)]$$
>
> **Assumption A2:** The class of online deterministic algorithms considered is finite.

The competitiveness of a deterministic algorithm $A$ under the distribution $P$ is defined by:

$$\mathcal{C}_A = \inf\left\{C : \sup_N \left(E_P[f_A(\Phi_N)] - CE_P[f_O(\Phi_N)]\right) < +\infty\right\}$$

and the oblivious competitiveness coefficient of a randomized algorithm $R$ is defined by:

$$\mathcal{C}_R^{obl} = \inf\left\{C : \sup_{N,\Phi_N} \left(E_R[f_R(\Phi_N)] - Cf_O(\Phi_N)\right) < +\infty\right\}$$

Prove the following bound:

$$\inf_R \mathcal{C}_R^{obl} \geq \inf_A \mathcal{C}_A^P \tag{1}$$

In practice, to use this bound, one selects a simple distribution $P$ so that it is possible to identify, or bound the performance of, the optimal deterministic algorithm, so that the RHS of (1) can be computed of bounded below. (Hint: Follow through the definitions and use the fact that a weighted average of a finite set of numbers is less than or equal to the maximum of the numbers.)

**2. On the Marker randomized cache eviction algorithm (Ramakrishna Gummadi)**
(a) Show that if the number of types is limited to $k+1$, then the Marker algorithm is $H_k$-competitive against oblivious adversaries. (Hint: Show that any algorithm, or the MIN algorithm, has at least $1 - d_I + d_F$ misses per time period.)
(b) Show that Marker is not $H_k$ competitive if the number of types is not restricted.

**3. Stopping time properties**
(a) Show that if $S$ and $T$ are stopping times for some filtration $\boldsymbol{\mathcal{F}}$, then $S \wedge T$, $S \vee T$, and $S + T$, are also stopping times.
(b) Show that if $\boldsymbol{\mathcal{F}}$ is a filtration and $X = (X_k : k \geq 0)$ is the random sequence defined by $X_k = I_{\{T\leq k\}}$ for some random time $T$ with values in $\mathbb{Z}_+$, then $T$ is a stopping time if and only if $X$ is $\boldsymbol{\mathcal{F}}$-adapted.
(c) If $T$ is a stopping time for a filtration $\boldsymbol{\mathcal{F}}$, recall that $\mathcal{F}_T$ is the set of events $A$ such that $A \cap \{T \leq n\} \in \mathcal{F}_n$ for all $n$. (Or, for discrete time, the set of events $A$ such that $A \cap \{T = n\} \in \mathcal{F}_n$ for all $n$.) Show that (i) $\mathcal{F}_T$ is a $\sigma$-algebra, (ii) $T$ is $\mathcal{F}_T$ measurable, and (iii) if $X$ is an adapted process then $X_T$ is $\mathcal{F}_T$ measurable.

**4. A stopped random walk**

Let $W_1, W_2, \ldots$ be a sequence of independent, identically distributed mean zero random variables. To avoid triviality, assume $P\{W_1 = 0\} \neq 0$. Let $S_0 = 0$ and $S_n = W_1 + \ldots W_n$ for $n \geq 1$. Fix a constant $c > 0$ and let $\tau = \min\{n \geq 0 : |S_n| \geq c\}$. The goal of this problem is to show that $E[S_\tau] = 0$.

(a) Show that $E[S_\tau] = 0$ if there is a constant $D$ so that $P[|W_i| > D] = 0$. (Hint: Invoke a version of the optional stopping theorem).

(b) In view of part (a), we need to address the case that the $W$'s are not bounded. Let $\widetilde{W}_n = \begin{cases} W_n & \text{if } |W_n| \leq 2c \\ a & \text{if } W_n > 2c \\ -b & \text{if } W_n < -2c \end{cases}$

where the constants $a$ and $b$ are selected so that $a \geq 2c$, $b \geq 2c$, and $E[\widetilde{W}_i] = 0$. Note that if $\tau < n$ and if $\tilde{W}_n \neq W_n$, then $\tau = n$. Thus, $\tau$ defined above also satisfies $\tau = \min\{n \geq 0 : |\widetilde{S}_n| \geq c\}$. Let $\widetilde{\sigma}^2 = \mathrm{Var}(\widetilde{W}_i)$. Let $\widetilde{S}_n = \widetilde{W}_1 + \ldots \widetilde{W}_n$ for $n \geq 0$ and let $M_n = \widetilde{S}_n^2 - n\widetilde{\sigma}^2$. Show that $M$ is a martingale. Hence, $E[M_{\tau \wedge n}] = 0$ for all $n$. Conclude that $E[\tau] < \infty$

(c) Show that $E[S_\tau] = 0$. (Hint: Use part (b) and invoke a version of the optional stopping theorem).

**5. Application of Foster's criteria (Yuksel Serdar)**

Consider the following model for two service stations in series. Let $0 < \lambda < \mu_1 < \mu_2 < 1$. Suppose $(A(k) : k \geq 1)$ is a sequence of $Be(\lambda)$ random variables, and for $i \in \{0, 1\}$, suppose $(S_i(k) : k \geq 1)$ is a sequence of $Be(\mu_i)$ random variables. All these random variables are assumed to be independent. The number of customers in station $i$ at time $k$ is denoted by $Q^i(k)$ for $i \in \{0, 1\}$. The overall system satisfies the evolution equation:

$$\begin{aligned} Q^1(k+1) &= Q^1(k) - u_1(k)S_1(k) + A_1(k+1) \\ Q^2(k+1) &= Q^2(k) + u_1(k)S_1(k) - u_2(k)S_2(k) \end{aligned}$$

where $u_i(k) = I_{\{Q^1(k) \geq 1\}}$. Some initial state $(q_0^1, q_0^2)$ is also given. Then, $Q = (Q(k) : k \geq 0)$, with $Q(k) = (Q^1(k), Q^2(k))$, is a time-homogeneous, discrete-time Markov process with state space $\mathbb{Z}_+^2$. Show that $Q$ is positive recurrent. (Hint: Try a quadratic potential function.)

**6. Bounding the value of a game**

Consider the following game. Initially a jar has $a_o$ red marbles and $b_o$ blue marbles. On each turn, the player removes a set of marbles, consisting of either one or two marbles of the same color, and then flips a fair coin. If heads appears on the coin, then if one marble was removed, one of each color is added to the jar, and if two marbles were removed, then three marbles of the other color are added back to the jar. If tails appears, no marbles are added back to the jar. The turn is then over. Play continues until the jar is empty after a turn, and then the game ends. Let $\tau$ be the number of turns in the game. The goal of the player is to minimize $E[\tau]$. A strategy is a rule to decide what set of marbles to remove at the beginning of each turn.

(a) Find a lower bound on $E[\tau]$ that holds no matter what strategy the player selects.

(b) Suggest a strategy that approximately minimizes $E[\tau]$, and for that strategy, find an upper bound on $E[\tau]$.