Chapter Six

NONLINEAR DC ANALYSIS

OR: Solution of Nonlinear Algebraic Equations

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Figure 6.3.1: A flowchart of DC solution algorithm.
Nonlinear Algebraic Equations

• A system of linear equations $Ax = b$ has a unique solution, unless $A$ is singular.

• However, a system of nonlinear equations $f(x) = y$ may have one solution, multiple finite solutions, no solution, or infinite number of solutions.
Example
Example
Example

\[ y = x^2 \] has two solutions for \( y > 0 \)
one solution for \( y = 0 \)
no solution for \( y < 0 \)
Example

\[ y = x^3 \] has a unique solution for every \( y \).
n-dimensional case

- \( f(x) = y \)

or,

\[
f_1(x_1, x_2, \ldots, x_n) = y_1 \\
f_2(x_1, x_2, \ldots, x_n) = y_2 \\
\vdots \\
f_n(x_1, x_2, \ldots, x_n) = y_n
\]
Problem

• Given $y \in \mathbb{R}^n$, find $x^* \in \mathbb{R}^n$, if it exists, such that $f(x^*) = y$.

• Some Theorems on the Existence and Uniqueness of Solutions of Nonlinear Resistive Networks.
Definition

• Given a mapping $f(.)$: $\mathbb{R}^n \rightarrow \mathbb{R}^n$. 

• $f \in C^1$ means $f$ is continuously differentiable; and

  $f$ is $C^1$ diffeomorphism means that the inverse function $f^{-1}$ exists, and is also of class $C^1$. 
Palais's Theorem

- The necessary and sufficient conditions that the mapping $f(.): \mathbb{R}^n \to \mathbb{R}^n$ to be a $C^1$ diffeomorphism of $\mathbb{R}^n$ onto itself are:

  (i) $f$ is of class $C^1$

  (ii) $\det [J] = \det \left[ \frac{\partial f}{\partial x} \right] \neq 0$

  (iii) $\lim ||f(x)|| \to \infty$ as $||x|| \to \infty$

- For existence and uniqueness of solution, can allow $\det [J] = 0$ at isolated points as long as it **does not change signs** and

  $\lim ||f(x)|| \to \infty$ when $||x|| \to \infty$
Circuit-Theoretic Theorems

- **Theorem 2** (Duffin): In a network consisting of independent voltage and current sources, and voltage-controlled two-terminal resistors \((i = g(v))\), there exists at least one solution provided that each resistor's \(v\)-\(i\) characteristic function \(g(v)\) is continuous in \(v\) and satisfies:

\[
g(v) \rightarrow +\infty \text{ (or } -\infty)\]

as \(v \rightarrow +\infty \text{ (or } -\infty)\)
Circuit-Theoretic Theorems (cont.)

- **Theorem 3** (Duffin): In a network consisting of independent voltage and current sources and voltage-controlled resistors \((i = g(v))\), there exists *at most one solution* provided that each resistor's \(v\) to \(i\) characteristic function \(g\) is strictly monotone increasing:

  \[
  g(v_1) < g(v_2) \quad \text{if} \quad v_1 < v_2
  \]

For existence and uniqueness, both Theorems 2 and 3 should be satisfied.
Circuit-Theoretic Theorems (cont.)

• **Theorem 3** (Desoer and Katzenelson): A **sufficient** condition for the existence of a unique solution for a network *consisting of time-varying voltage-controlled* and current-controlled resistors characterized by **continuous** (not necessarily strictly) monotone increasing functions, and *independent voltage* and *current sources*, is that the resistor network formed by *short-circuiting all voltage sources* and *open-circuiting all current sources* has a **tree** (or forest) such that all tree branches correspond to *current-controlled elements* and all links correspond to *voltage-controlled elements*. 
Now back to the numerical solution of:

\[ f(x) = y \]

Given \( y \), find \( x \) (assuming it exists)
Fixed-Point Iteration

\[ x = g(x) \]

• A given problem can be recast into fixed-point problem, where \( x = g(x) \) is a suitably chosen function whose solutions are the solution of \( f(x) = y \).

• For example, \( f(x) = y \) can be written as 
  \[ x = f(x) - y + x = g(x). \]

• Given \( x = g(x) \)

• Fixed-Point Iteration: 
  \[ x^{k+1} = g(x^k) \]
  Repeat until \( \|x^{k+1} - x^{k}\| < \varepsilon \)
Examples

\[ |\frac{dg}{dx}| < 1 \]

converges
\[
\frac{|\frac{dg}{dx}|}{g(x)} > 1
\]

\[
x = x^0
\]

\[
\text{diverges}
\]
Examples (cont.)

Multiple solutions
Contraction mapping theorem

• Suppose \( g: D \subset \mathbb{R}^n \rightarrow \mathbb{R}^n \) maps a closed set \( D_0 \subset D \) into itself and \( ||g(x) - g(y)|| \leq \alpha ||x-y|| \), \( x, y \in D_0 \) for some \( \alpha < 1 \)

Then for any \( x^0 \in D_0 \), the sequence \( x^{k+1} = g(x^k) \), \( k = 0, 1, 2, \ldots \), converges to a unique \( x^* \) of \( g \) in \( D_0 \).

• Proof:

\[
||x^*-x^k|| = ||g(x^*) - g(x^{k-1})|| \\
\leq \alpha ||x^*-x^{k-1}|| \\
\leq \alpha^k ||x^*-x^0||
\]

Since \( \alpha < 1 \), \( \alpha^k \rightarrow 0 \) and \( ||x^*-x^k|| \rightarrow 0 \) or \( x^k \rightarrow x^* \)
Parallel Chord Method

\( \mathbf{A} \) remains constant; it is usually chosen to be 
\[
\begin{bmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n}
\end{bmatrix}
\]
where \( \frac{\partial f}{\partial \mathbf{x}} \)

Instead of computing \( \mathbf{A}^{-1} \), the following equation is solved:

\[
\mathbf{A}(\mathbf{x}^{k+1} - \mathbf{x}^k) = \mathbf{y} - \mathbf{f}(\mathbf{x}^k) \quad \text{or} \quad \mathbf{A} \Delta \mathbf{x}^k = \Delta \mathbf{y}^k
\]

At every iteration, \( \Delta \mathbf{y}^k \) changes, while \( \mathbf{A} \) (and its LU factors) remain unchanged.
Parallel Chord Method

\[ x^{k+1} = x^k + A^{-1}(y - f(x^k)) \]

Or

\[ A(x^{k+1} - x^k) = (y - f(x^k)) \]
Algorithm 6.2.2: Parallel Chord Iteration Algorithm

Given \( f(x) = y \) with a specified \( y \).

1. Start with an initial guess \( x^{(o)} \)
2. Construct \( A = \left[ \frac{\partial f}{\partial x} \right] \) evaluated at \( x = x^{(o)} \); or choose any other appropriate matrix \( A \)
3. For \( k = 0, 1, 2, \ldots \)
4. Find \( \Delta y^{(k)} = y - f(x^{(k)}) \)
5. Solve \( A \Delta x^{(k)} = \Delta y^{(k)} \)
6. Put \( x^{(k)} = x^{(k)} + \Delta x^{(k)} \)
7. Repeat until \( \| \Delta x^{(k)} \| < \varepsilon \) and \( \| \Delta y^{(k)} \| < \delta \), where \( \varepsilon \) and \( \delta \) are specified small positive numbers.
Parallel Chord Method
Parallel Chord Example (nonconvergence)
Newton's (or Newton-Raphson) Method

Given: \( f(x) = y \)

\[
f(x) = f(x^k) + \left[ \frac{\partial f}{\partial x} \right]_{x=x^k} (x - x^k) = y
\]

First two-terms of Taylor Series expansion

OR

\[
[J^k](x-x^k) = (y-f(x^k))
\]

\[
[J^k] \Delta x^k = \Delta y^k \quad \text{(solve)}
\]

\[
x^{k+1} = x^k + \Delta x^k
\]

\[
x^{k+1} = x^k + \left[ J^k \right]^{-1} \left( y - f(x^k) \right) = g(x^k)
\]

where \( J^k \equiv \left[ \frac{\partial f}{\partial x} \right]_{x=x^k} \)
Newton or Newton-Raphson Algorithm

Given $f(x) = y$ with a specified $y$
1. Choose an initial guess $x^{(o)}$
2. For $k = 0, 1, 2, \ldots$
3. Find $\Delta y^{(k)} = y - f(x^{(k)})$
4. Construct $[J^{(k)}] = \left[ \frac{\partial f}{\partial x} \right]$ evaluated at $x = x^{(k)}$
5. Solve $[J^{(k)}] \Delta x^{(k)} = \Delta y^{(k)}$
6. Put $x^{(k+1)} = x^{(k)} + \Delta x^{(k)}$
7. Repeat until $\| \Delta x^{(k)} \| < \varepsilon$ and $\| \Delta y^{(k)} \| < \delta$,
Newton's (or Newton-Raphson) Method

The slope changes with every iteration.
Convergence Properties of Newton's Method

• Applying Taylor Series expansion at \( x^k \):

\[
y = f(x^*) = f(x^k) + J^k(x^* - x^k) + R(x^* - x^k)
\]

where \( x^* \) is the solution

• If the derivative of \( J^k \) (i.e., second derivative of \( f \)) is bounded, then:

\[
||R(x^k - x^*)|| \leq \alpha ||x^k - x^*||^2
\]

• Newton's Method:

\[
x^{k+1} = x^k + [J^k]^{-1} (y - f(x^k))
\]

or,

\[
[J^k](x^{k+1} - x^k) = y - f(x^k)
\]
n-dimensional case (cont.)

From Taylor Series:

• \( y - f(x^k) = J^k(x^* - x^k) + R(x^* - x^k) \)

• \( J^k(x^{k+1} - x^k) = J^k(x^* - x^k) + R(x^* - x^k) \)

• \( x^{k+1} - x^k = x^* - x^k + [J^k]^{-1}R(x^* - x^k) \)

• \( x^{k+1} - x^* = [J^k]^{-1}R(x^* - x^k) \)

• \( ||x^{k+1} - x^*|| \leq c \ ||x^* - x^k||^2 \)

Provided \( J(x^*) \) is nonsingular
Rate of Convergence

- Define $e_k = x^* - x^k$

- A method is said to converge with rate $r$ if:
  
  $$||e_{k+1}|| = c ||e_k||^r$$

  for some nonzero constant $c$

- If $r = 1$, the convergence is linear.

- If $r > 1$, the convergence rate is superlinear.

- If $r = 2$, the convergence rate is quadratic.
Newton's Method

Has \textit{quadratic convergence} if \( x^k \) is "close enough" to the \textbf{solution} and \( J(x^*) \) is \textit{nonsingular}. 
Convergence Problems of Newton’s Method
Convergence Problems of Newton’s Method
Pseudo-Newton or Norm-Reducing Techniques

(1) In the Newton algorithm the Jacobian is constructed and factorized at every iteration, while in the parallel-chord the Jacobian is constructed and factorized once, and only the forward and backward substitutions are applied at every iteration. One strategy to reduce computation is to apply the parallel-chord algorithm as long as \( \|y - y^{(k+1)}\| < \|y - y^{(k)}\| \) and switch to Newton method when (i) \( \|y - y^{(k+1)}\| \geq \|y - y^{(k)}\| \); (ii) when the iterations approach the solution, that is, \( \|y - y^{(k+1)}\| < \sigma \), for some \( \sigma > \delta \); or (iii) when convergence slows down, since Newton algorithm has quadratic convergence in the vicinity of the solution, while the parallel-chord method has linear convergence.
(2) In Newton algorithm, after solving \( [J^{(k)}] \Delta x^{(k)} = \Delta y^{(k)} \), put

\[
x^{(k+1)} = x^{(k)} + \lambda \Delta x^{(k)}
\]

where \( \lambda \) is chosen such that \( \|y - y^{(k+1)}\| < \|y - y^{(k)}\| \)

This normally requires a line search along the vector \( x^{(k)} + \lambda \Delta x^{(k)} \)
(3) After solving $[J^{(k)}] \Delta x^{(k)} = \Delta y^{(k)}$, put

$$x^{(k+1)} = x^{(k)} + \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix} \Delta x^{(k)}$$
(4) When solving $[J^{(k)}] \Delta x^{(k)} = \Delta y^{(k)}$, if $J^{(k)}$ is singular or ill-conditioned, or if a zero or a small pivot is encountered and sparse matrix reordering is not desirable, one can solve a modified system

$$
\begin{bmatrix}
J^{(k)} + \\
\lambda_1 \\
\lambda_2 \\
\ldots \\
\lambda_n
\end{bmatrix} \Delta x^{(k)} = \Delta y^{(k)}
$$

Normally, very few of the $\lambda_i$s are nonzero. To save on computation, it is possible to correct the solution of (6.2.17) to get the solution of the original equation $[J^{(k)}] \Delta x^{(k)} = \Delta y^{(k)}$ by using the solution updating algorithm
(5) Source Stepping: If \( f(x) = 0 \) is easy to solve, or if it is readily available, solve

\[
f(x) = \lambda y
\]

for \( \lambda = 0, \lambda_1, \lambda_2, \ldots, 1 \), where \( 0 < \lambda_i < \lambda_{i+1} < 1 \). The solution of the equations at \( \lambda_i \) is used as the initial guess for the solution at \( \lambda_{i+1} \). When \( \lambda = 1 \), \( f(x) = y \). The philosophy behind this strategy is that the solution at \( \lambda_{i+1} \) is close to the solution at \( \lambda_i \), if the increment in \( \lambda \) is small enough. In circuit application, source stepping is equivalent to putting initially all the independent sources to zero, which gives a zero solution, and then increasing the values of the independent sources in increments until the given values are reached.
(6) Homotopy: Solve

\[ \lambda f(x) + (1 - \lambda)(x - x^{(o)}) = \lambda y \]

for \( \lambda = 0, \lambda_1, \lambda_2, \ldots, 1 \), where \( 0 < \lambda_i < \lambda_{i+1} < 1 \). The solution of the equations at \( \lambda_i \) is used as the initial guess for the solution at \( \lambda_{i+1} \). When \( \lambda = 0 \), \( x = x^{(o)} \); when \( \lambda = 1 \), \( f(x) = y \). Again, the philosophy behind this strategy is that the solution at \( \lambda_{i+1} \) is close to the solution at \( \lambda_i \), if the increment in \( \lambda \) is small enough.
(7) Approximate \( f(x) \) by another close function, \( \hat{f} \), that is easy to solve, such as a piecewise-linear function, then use the solution of \( \hat{f}(x) = y \) as the initial solution to solve \( f(x) = y \)

Piecewise-linear solution techniques are covered in Chapter 7
(8) Form a differential equation:

$$\dot{x} + f(x) = yu(t)$$

where $\dot{x} \equiv \frac{dx}{dt}$, and $u(t)$ is the step function. Assume the initial condition at $f(x) = 0$ can be easily found (when all inputs are put to zero, normally $x = 0$). When $t \to \infty$, the steady-state solution of (6.2.20) approaches the solution of

$$f(x) = y$$

The numerical solution of differential equations is covered in Chapter 8.
Application to Electronic Circuits:

- Capacitors are open and inductors are short-circuited. Why?

Tableau Equations:

\[ \text{KCL: } A_i = 0 \text{ (or } Q_i = 0) \]
\[ \text{KVL: } v_b = A^T v_n \text{ (or } B v_b = 0) \]
\[ i_1 = f_1(v_1, i_2) + s_1 \]
\[ v_2 = f_2(v_1, i_2) + s_2 \]

In general, the characteristics could be of the form:

\[ f(v_b, i_b, z) = 0 \]
there is no need to formulate the nonlinear MNA equations since the solution is the same regardless of the formulation method used

The nonlinear elements are linearized at the tableau equation level

that is, at the element characteristics level

The linearized circuit equations are then formulated and solved using the MNA formulation or any other formulation method
DC Solution Algorithm

1. Read the circuit topology specifications and the element characteristics.

2. Specify an initial solution guess, \( x^{(k)} \), \( k = 0 \). If the modified nodal analysis (MNA) formulation is used, then \( x = [ v_n \quad i_2 ]^T \); if the extended nodal analysis (ENA) formulation is used, then \( x = [ v_n \quad i_2 \quad z ]^T \).

3. Linearize the nonlinear element characteristics at \( x = x^{(k)} \).

4. Formulate the linearized circuit equations using any general formulation method, such as the MNA stamp formulation method, to get the linearized circuit equation either in the form

\[
J^{(k)} x = b^{(k)}, \quad \text{where} \quad b^{(k)} = y - f(x^{(k)}) + [J^{(k)}] x^{(k)}
\]

or as

\[
J^{(k)} \Delta x^{(k)} = \Delta y^{(k)}, \quad \text{where} \quad \Delta y^{(k)} = y - f(x^{(k)})
\]
5. Solve the linearized circuit equations, $J^{(k)}x = b^{(k)}$ or $J^{(k)}\Delta x^{(k)} = \Delta y^{(k)}$, normally by using LU factorization and sparse matrix techniques, or by an iterative method, to obtain $x^{(k+1)}$ or $\Delta x^{(k)}$. The structure of $J^{(k)}$ does not normally change from one iteration to the next, but the values of its entries may change.

6. **Check for convergence:** Check if $\|\Delta x^{(k)}\| = \|x^{(k)} - x^{(k+1)}\| < \epsilon$ (in circuit simulation normally the $L_\infty$-norm is used). If not, go to step 3 and repeat. If $\|\Delta x^{(k)}\| < \epsilon$, check if $\|\Delta y^{(k+1)}\| = \|y - y^{(k+1)}\| < \delta$. If it is, a solution is found. If not, a *false* solution is reached, and a new initial point should be chosen, or a modified Newton method applied (see Figure 6.2.9). In some applications, $\Delta y^{(k+1)}$ is not computed to save on computation, and a solution is declared when $\|\Delta x^{(k)}\| < \epsilon$, even though a solution is not guaranteed to have been reached.
Choose an initial guess $x^{(k)}$, $k=0$

Linearize nonlinear elements at $x^{(k)}$

$k \rightarrow k+1$

Construct and solve linearized circuit equations $Ax=b$

Check for convergence

No

Yes

A solution is found
- In step (5) above, a relaxation method may be applied to solve $Ax = b$ rather than LU factorization.

- If the parallel-chord method is applied, only the right-hand side vector $b$ is constructed at the next iteration and the LU factorization is not performed in step 5; only the forward and backward substitution steps are repeated.

- In step 3, only the nonlinear resistors in which the independent variables did not converge are re-linearized.
• Modified Newton methods described in the previous section can be made part of the above algorithm without much difficulty.

• It is possible to mix the parallel-chord method with the Newton method. For example, one could use the parallel-chord method as long as the iterates are converging in order to save on formulating the linearized equations and performing LU factorization at every iteration, then switch to Newton method when the iterates approach the solution, that is, $\|\Delta x^{(k)}\|$ is less than a certain threshold, or when the iterates diverge.
Stamps: Nonlinear Elements

Two-terminal voltage-controlled resistor:

\[ i = g(v) \]

Taylor series expansion at an iteration point \( v^{(k)} \)

\[ i = g(v^{(k)}) + \left[ \frac{dg}{dv} \right]_{v=v^{(k)}} (v - v^{(k)}) = a^{(k)}v + g(v^{(k)}) - a^{(k)}v^{(k)} \]

\[ i = a^{(k)}v + b^{(k)} \]
\[ g(v^{(k)}) \]

\[ v^{(k)} \]

\[ b^{(k)} \]

\[ a^{(k)} \]

\[
\begin{bmatrix}
    n_1 & +a^{(k)} & n_2 & -a^{(k)} & \cdots \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    n_2 & -a^{(k)} & n_1 & +a^{(k)} & \cdots \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
    \vdots & \ddots & \vdots & \ddots & \vdots \\
\end{bmatrix}
\begin{bmatrix}
    v_{n1} \\
    v_{n2} \\
\end{bmatrix}
=
\begin{bmatrix}
    rhs \\
    -b^{(k)} \\
    +b^{(k)} \\
\end{bmatrix}
\]
Two-terminal current-controlled resistor:

\[ v = r(i) \]

\[ v = r(i^{(k)}) + \left[ \frac{dr}{di} \right]_{i=i^{(k)}} \quad (i - i^{(k)}) = a^{(k)} i + r(i^{(k)}) - a^{(k)} i^{(k)} \]

\[ v = a^{(k)} i + b^{(k)} \]
Two-terminal current-controlled resistor:

\[
\begin{bmatrix}
  \vdots & n_1 & \vdots & n_2 & \vdots & +1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & -1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & -a^{(k)} \\
  +1 & -1 & -a^{(k)} & & & \\
\end{bmatrix}
\begin{bmatrix}
  v_{n1} \\
  v_{n2} \\
  i \\
  +b^{(k)} \\
\end{bmatrix}
= \begin{bmatrix}
  rhs \\
\end{bmatrix}
\]
Three-terminal voltage-controlled resistor:

\[ i_1 = f_1(v_1, v_2) \]

\[ i_2 = f_2(v_1, v_2) \]
Taylor series expansion

\[ i_1 = a_{11}^{(k)} v_1 + a_{12}^{(k)} v_2 + b_1^{(k)} \]

\[ i_2 = a_{21}^{(k)} v_1 + a_{22}^{(k)} v_2 + b_2^{(k)} \]

\[ a_{11}^{(k)} = \frac{\partial f_1}{\partial v_1}, \quad a_{12}^{(k)} = \frac{\partial f_1}{\partial v_2}, \quad a_{21}^{(k)} = \frac{\partial f_2}{\partial v_1}, \quad a_{22}^{(k)} = \frac{\partial f_2}{\partial v_2} \]

evaluated at \( v_1 = v_1^{(k)}, \ v_2 = v_2^{(k)} \)

\[ b_1^{(k)} = f_1(v_1^{(k)}, v_2^{(k)}) - (a_{11}^{(k)} v_1^{(k)} + a_{12}^{(k)} v_2^{(k)}) \]

\[ b_2^{(k)} = f_2(v_1^{(k)}, v_2^{(k)}) - (a_{21}^{(k)} v_1^{(k)} + a_{22}^{(k)} v_2^{(k)}) \]
Three-terminal voltage-controlled resistor:

\[
\begin{bmatrix}
  n_1 & n_2 & n_3 \\
  +a_{11}^{(k)} & +a_{12}^{(k)} & -(a_{11}^{(k)} + a_{12}^{(k)}) \\
  +a_{21}^{(k)} & +a_{22}^{(k)} & -(a_{21}^{(k)} + a_{22}^{(k)}) \\
  -(a_{11}^{(k)} + a_{21}^{(k)}) & -(a_{12}^{(k)} + a_{22}^{(k)}) & a_{33}^{(k)}
\end{bmatrix}
\begin{bmatrix}
  v_{n1} \\
  v_{n2} \\
  v_{n3}
\end{bmatrix}
= \begin{bmatrix}
  r_{\text{rhs}} \\
  -b_1^{(k)} \\
  -b_2^{(k)} \\
  +b_1^{(k)} + b_2^{(k)}
\end{bmatrix}
\]

where \( a_{33}^{(k)} = (a_{11}^{(k)} + a_{21}^{(k)} + a_{12}^{(k)} + a_{22}^{(k)}) \)
Three-terminal resistor with hybrid characteristics:

\[ i_1 = f_1(v_1, i_2) \]
\[ v_2 = f_2(v_1, i_2) \]

Applying Taylor series expansion

\[ i_1 = a_{11}^{(k)} v_1 + a_{12}^{(k)} i_2 + b_1^{(k)} \]
\[ v_2 = a_{21}^{(k)} v_1 + a_{22}^{(k)} i_2 + b_2^{(k)} \]
Three-terminal resistor with hybrid characteristics:

\[
\begin{bmatrix}
  n_1 & n_2 & n_3 \\
  +a^{(k)}_{11} & -a^{(k)}_{11} & : & +a^{(k)}_{12} \\
  -a^{(k)}_{11} & +a^{(k)}_{11} & : & (1 - a^{(k)}_{12}) \\
  \cdots & \cdots & \cdots & \cdots \\
  -a^{(k)}_{21} & 1 & (-1 + a^{(k)}_{21}) & : & -a^{(k)}_{22}
\end{bmatrix}
\begin{bmatrix}
v_{n1} \\
v_{n2} \\
v_{n3} \\
i_2
\end{bmatrix}
=
\begin{bmatrix}
\text{rhs} \\
-b^{(k)}_1 \\
+b^{(k)}_1 \\
+b^{(k)}_2
\end{bmatrix}
\]
Other methods of linearization

Approximate differentiation:

\[
\frac{df}{dx} \approx \frac{f(x^{(k)} + \Delta x^{(k)}) - f(x^{(k)})}{\Delta x^{(k)}} = a^{(k)}
\]

\[
y = a^{(k)} x + b^{(k)} \quad b^{(k)} = f(x^{(k)}) - a^{(k)} x^{(k)}
\]

\[
y = f(x)
\]

\[
\frac{\partial f_i}{\partial x_j} \approx \frac{f_i(x_1^{(k)}, \ldots, x_j^{(k)} + \Delta x_j^{(k)}, \ldots, x_n^{(k)}) - f_i(x^{(k)})}{\Delta x_j^{(k)}}
\]

\[
y = J^{(k)} x + b^{(k)}, \quad b^{(k)} = (f(x^{(k)}) - J^{(k)} x^{(k)})
\]
(2) Secant method:

\[
\frac{df}{dx} \approx \frac{f(x^{(k)}) - f(x^{(k-1)})}{x^{k} - x^{k-1}} = a^{(k)}
\]

\[
\frac{\partial f_i}{\partial x_j} \approx \frac{f_i(x^{(k)}) - f_i(x_1^{(k)}, \ldots, x_j^{(k)}, \ldots, x_n^{(k)})}{x_j^{(k)} - x_j^{(k-1)}}
\]
(3) Line-through-the-origin:

\[ a^{(k)} = \frac{f(x^{(k)})}{x^{(k)}} \text{ and } b^{(k)} = 0, \]

n-dimensional case \[ \frac{\partial f_i}{\partial x_j} \approx f_i(x^{(k)})/x_j, \text{ and } b^{(k)} = 0 \]
Diode:

The characteristic equation of a diode, not including breakdown

\[ i_d = g(v_d) = I_s \left( e^{v_d/\eta V_T} - 1 \right) \]

typical value for \( I_s \) is \( 10^{-14} \text{A} \)

\( V_T \approx 26 \text{ mV at } 300^\circ \text{K} \)

Taylor series expansion an iteration point \( v_d^k \)

\[ i_d = a^{(k)} v_d + b^{(k)} \]

\[ a^{(k)} = \frac{I_s}{\eta V_T} e^{v_d^{(k)}/\eta V_T} \text{ and } b^{(k)} = g(v_d^{(k)}) - a^{(k)} v_d^{(k)} \]
if $v_d^{(k)} = 5V$ \[ g(v_d^{(k)}) \approx 3.272 \times 10^{69} \text{ and } a^{(k)} \approx 1.2585 \times 10^{72} \]

Voltage-Current Iteration Strategy

![Graph depicting voltage-current iteration strategy]
Newton iteration produced a new iteration point

\[ \hat{v}_d^{(k+1)} = v_d^{(k)} + \Delta v_d^{(k)} \]

(a) If \( \hat{v}_d^{(k+1)} \leq v_d^{crit} \)

\[ v_d^{(k+1)} = \hat{v}_d^{(k+1)} \]

(b) If \( \hat{v}_d^{(k+1)} > v_d^{crit} \)

\[ \hat{i}_d^{(k+1)} = a^{(k)} \hat{v}_d^{(k+1)} + b^{(k)} \]

\[ \hat{i}_d^{(k+1)} = I_s \left( e^{v_d^{(k+1)}} / \eta V_T - 1 \right), \text{ or} \]

\[ v_d^{(k+1)} = \eta V_T \ln \left( 1 + \frac{\hat{i}_d^{(k+1)}}{I_s} \right) \]
\[ v_{d}^{\text{crit}} = \eta V_T \ln\left( \frac{\eta V_T}{\sqrt{2} I_s} \right) \]

\[ v_{d}^{\text{crit}} \] is the value of \( v_d \) where the radius of curvature of the diode characteristic equation is minimum.

If \( \eta V_T = 26 \text{ mV} \) and \( I_s = 10^{-14} \), then \( v_{d}^{\text{crit}} \approx 0.734 \text{V} \)
NPN Bipolar Junction Transistor: Ebers-Moll Model

\[ i_c = -i_r + \alpha_f i_f = -I_{cs} \left( e^{\frac{v_{bc}}{V_{Tc}}} - 1 \right) + \alpha_f I_{es} \left( e^{\frac{v_{be}}{V_{Te}}} - 1 \right) \]

\[ i_e = -i_f + \alpha_r i_r = \alpha_r I_{cs} \left( e^{\frac{v_{bc}}{V_{Tc}}} - 1 \right) - I_{es} \left( e^{\frac{v_{be}}{V_{Te}}} - 1 \right) \]
Taylor series expansion at iteration point $v_{bc}^{(k)}$ and $v_{be}^{(k)}$

\[ i_c = a_{11}^{(k)} v_{bc} + a_{12}^{(k)} v_{be} + b_1^{(k)} \]

\[ i_e = a_{21}^{(k)} v_{bc} + a_{22}^{(k)} v_{be} + b_2^{(k)} \]

\[ a_{11}^{(k)} = -\frac{I_{cs}}{V_{Tc}} \left( e^{v_{bc}^{(k)}} / V_{Tc} \right); \quad a_{12}^{(k)} = \frac{\alpha_f I_{es}}{V_{Te}} \left( e^{v_{be}^{(k)}} / V_{Te} \right) \]

\[ b_1^{(k)} = -I_{cs} \left( e^{v_{bc}^{(k)}} / V_{Tc} - 1 \right) + \alpha_f I_{es} \left( e^{v_{be}^{(k)}} / V_{Te} - 1 \right) - a_{11}^{(k)} v_{bc}^{(k)} - a_{12}^{(k)} v_{be}^{(k)} \]

\[ a_{21}^{(k)} = \frac{\alpha_r I_{cs}}{V_{Tc}} \left( e^{v_{bc}^{(k)}} / V_{Tc} \right); \quad a_{22}^{(k)} = -\frac{I_{es}}{V_{Te}} \left( e^{v_{be}^{(k)}} / V_{Te} \right); \text{ and} \]

\[ b_2^{(k)} = \alpha_r I_{cs} \left( e^{v_{bc}^{(k)}} / V_{Tc} - 1 \right) - I_{es} \left( e^{v_{be}^{(k)}} / V_{Te} - 1 \right) - a_{21}^{(k)} v_{bc}^{(k)} - a_{22}^{(k)} v_{be}^{(k)}. \]
\[
\begin{bmatrix}
-a_{11}^{(k)} & -a_{12}^{(k)} & (a_{11}^{(k)} + a_{12}^{(k)}) \\
-a_{21}^{(k)} & -a_{22}^{(k)} & (a_{21}^{(k)} + a_{22}^{(k)}) \\
(a_{11}^{(k)} + a_{21}^{(k)}) & (a_{12}^{(k)} + a_{22}^{(k)}) & -a_{33}^{(k)}
\end{bmatrix}
\begin{bmatrix}
v_c \\
v_e \\
v_b
\end{bmatrix}
= \begin{bmatrix}
-b_1^{(k)} \\
-b_2^{(k)} \\
+b_1^{(k)} + b_2^{(k)}
\end{bmatrix}
\]

\[a_{33}^{(k)} = (a_{11}^{(k)} + a_{21}^{(k)} + a_{12}^{(k)} + a_{22}^{(k)})\]
PNP Bipolar Junction Transistor Model
MOSFET - NMOS

Diagram showing the Drain (D), Gate (G), Source (S), and Body (B) terminals of an NMOS transistor. The diagram also illustrates the relationship between Drain Current (I_DS) and Drain-Source Voltage (V_DS) with different Gate-Source Voltage (V_GS) values, indicating linear and saturation regions.
(i) **Subthreshold** (also known as weak-inversion or cut-off) region:

\[ V_{GS} < V_{th} \]

\[ I_D = I_D^0 e^{V_{ov}/\eta V_T} \]

\[ V_{ov} = V_{GS} - V_{th} \]

*\( V_{th} \) is the threshold voltage
*\( V_{GS} \) is the gate to source voltage

\[ V_{th} = V_{T0} + \gamma (\sqrt{V_{SB}} + 2\phi - \sqrt{2\phi}) \]
(ii) **Linear region:**

$$V_{GS} \geq V_{th} \text{ and } V_{DS} < V_{ov}.$$  

$$I_D = k_n[V_{ov}V_{DS} - \frac{V_{DS}^2}{2}]$$

$V_{DS}$ is the drain to source voltage  

$k_n = \mu_n C_{ox} \frac{W}{L}$  

$\mu_n$ is the charge-carrier effective mobility  

$W$ the gate width, $L$ the gate length.  

$C_{ox}$ the gate oxide capacitance per unit area
(iii) **Saturation or active region:** \( V_{GS} \geq V_{th} \) and \( V_{DS} > V_{ov} \)

\[
I_D = \frac{1}{2} k_n V_{ov}^2 [1 + \lambda (V_{DS} - V_{ov})]
\]

\( \lambda \) is a channel length modulation parameter.

when \( V_{DS} = V_{ov} \)

\[
k_n [V_{ov} V_{DS} - \frac{V_{DS}^2}{2}] = \frac{1}{2} k_n V_{ov}^2 [1 + \lambda (V_{DS} - V_{ov})]
\]

continuity of the equations across the regions
Taylor series expansion at iteration point $V_{GS}^{(k)}$ and $V_{DS}^{(k)}$

$$I = a_{11}^{(k)} V_{GS} + a_{12}^{(k)} V_{DS} + b_1^{(k)}$$

The coefficients are different in each region.

If $V_{SB} \neq 0$, $V_{th}$ is evaluated at the beginning of every iteration.
Taylor series expansion at $V_{GS}^{(k)}$, $V_{DS}^{(k)}$ and $V_{th}^{(k)}$

\[
I_D = a_{11}^{(k)} V_{GS} + a_{12}^{(k)} V_{DS} + a_{13}^{(k)} V_{th} + b_1^{(k)}
\]

\[
V_{th} = a_{21}^{(k)} V_{SB} + b_2^{(k)}
\]
Input-Output DC Characteristics

[Diagram of a nonlinear resistive circuit with input $v_{in}$ and output $v_{out}$]

[Graph showing the input-output characteristic with points marked at various inputs and outputs]
Constructing $J^{(k)} \triangle x^{(k)} = \triangle y^{(k)}$

**Independent current source:**

$$\triangle y^{(k)}(n_1) \leftrightarrow \triangle y^{(k)}(n_1) - I$$

$$\triangle y^{(k)}(n_2) \leftrightarrow \triangle y^{(k)}(n_2) + I$$

**Independent voltage source:**

$$
\begin{bmatrix}
  \ldots & n_1 & \ldots & n_2 & \ldots & +1 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \ldots & \ldots & \ldots & -1 & \vdots & \vdots \\
  +1 & -1 & 0 & & & \\
\end{bmatrix}
\begin{bmatrix}
  v_{n1} \\
  v_{n2} \\
  i_v \\
\end{bmatrix}
= 
\begin{bmatrix}
  rhs \\
  E \\
\end{bmatrix}
$$
Independent voltage source:

\[ \triangle y^{(k)}(n_1) \Leftarrow \triangle y^{(k)}(n_1) - i_v^{(k)} \]
\[ \triangle y^{(k)}(n_2) \Leftarrow \triangle y^{(k)}(n_2) + i_v^{(k)} \]
\[ \triangle y^{(k)}(p) \Leftarrow \triangle y^{(k)}(p) + E - (v_{n1}^{(k)} - v_{n2}^{(k)}) \]

after the initial solution in the Newton method, \( k \geq 1, E - (v_{n1}^{(k)} - v_{n2}^{(k)}) = 0. \)
Voltage-controlled resistor:

\[ i^{(k)} = f(v^{(k)}) \]

\[ \Delta y^{(k)}(n_1) \leftrightarrow \Delta y^{(k)}(n_1) - i^{(k)} \]

\[ \Delta y^{(k)}(n_2) \leftrightarrow \Delta y^{(k)}(n_2) + i^{(k)} \]
Current-controlled resistor:

\[
\begin{bmatrix}
  n_1 & n_2 \\
  \vdots & \vdots \\
  \vdots & \vdots \\
  \vdots & \vdots \\
  +1 & -1 & -r
\end{bmatrix}
\begin{bmatrix}
  v_{n1} \\
  v_{n2} \\
  i_r
\end{bmatrix} = \begin{bmatrix}
  rhs
\end{bmatrix}
\]

\[
\triangle y^{(k)}(n_1) \iff \triangle y^{(k)}(n_1) - i_r^{(k)}
\]

\[
\triangle y^{(k)}(n_2) \iff \triangle y^{(k)}(n_2) + i_r^{(k)}
\]

\[
\triangle y^{(k)}(p) \iff \triangle y^{(k)}(p) - (v_{n1}^{(k)} - v_{n2}^{(k)}) + v_r^{(k)}
\]
Three-terminal voltage-controlled resistor:

Let \( i_1^{(k)} = a_{11}v_1^{(k)} + a_{12}v_2^{(k)} \), \( i_2^{(k)} = a_{21}v_1^{(k)} + a_{22}v_2^{(k)} \) in the linear case,

\[ i_1^{(k)} = f_1(v_1^{(k)}, v_2^{(k)}) \] and \( i_2^{(k)} = f_2(v_1^{(k)}, v_2^{(k)}) \) in the nonlinear case.

\[
\begin{align*}
\triangle y^{(k)}(n_1) & \iff \triangle y^{(k)}(n_1) - i_1^{(k)} \\
\triangle y^{(k)}(n_2) & \iff \triangle y^{(k)}(n_2) - i_2^{(k)} \\
\triangle y^{(k)}(n_3) & \iff \triangle y^{(k)}(n_3) + i_1^{(k)} + i_2^{(k)}
\end{align*}
\]
Partitioning and Relaxation Methods

Partitioning into Linear and Nonlinear Subcircuits

\[
\begin{bmatrix}
A_{11} & 0 & A_{1t} \\
0 & A_{22}^{(k)} & A_{2t}^{(k)} \\
A_{t1} & A_{t2}^{(k)} & A_{tt}^{(k)}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_t
\end{bmatrix}
=
\begin{bmatrix}
b_1^{(k)} \\
b_2^{(k)} \\
b_t^{(k)}
\end{bmatrix}
\]
\[
\begin{bmatrix}
I & 0 & A'_{1t} \\
0 & A^{(k)}_{22} & A^{(k)}_{2t} \\
0 & A^{(k)}_{t2} & (A^{(k)}_{tt} - H_1)
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_t
\end{bmatrix}
= 
\begin{bmatrix}
b'_1 \\
b^{(k)}_2 \\
b^{(k)}_t - b'_1
\end{bmatrix}
\]

\[
\begin{bmatrix}
A^{(k)}_{22} & A^{(k)}_{2t} \\
A^{(k)}_{t2} & (A^{(k)}_{tt} - H_1)
\end{bmatrix}
\begin{bmatrix}
x_2 \\
x_t
\end{bmatrix}
= 
\begin{bmatrix}
b^{(k)}_2 \\
b^{(k)}_t - b'_1
\end{bmatrix}
\]
Relaxation Methods
Newton-Gauss-Jacobi and Newton-Gauss-Seidel

Choose an initial guess $x^{(k)} = 0$

Linearize nonlinear elements at $x^{(k)}$ and construct $Ax = b$

$k \rightarrow k + 1$

Solve $Ax = b$ by a relaxation method (G-J, G-S, or other)

Check for convergence

No

Yes

A solution is found
Gauss-Jacobi-Newton and Gauss-Seidel-Newton

Partition \( f(x) = y \) into
\( f_i(x_1, \ldots, x_i, \ldots, x_n) = y_i \)

Initial guess \( x^{(k)}, k = 0 \)

\[ f_i(x_1^{(k)}, \ldots, x_i^{(k)}, \ldots, x_n^{(k)}) = y_i \]

or
\[ f_i(x_1^{(k+1)}, \ldots, x_i^{(k+1)}, \ldots, x_n^{(k)}) = y_i \]

\( k \rightarrow k + 1 \)

Check for convergence

No

Yes

A solution is found
Sensitivity Analysis

\[ f(x, p) = y \]

\[ \psi = e^T x \]

\[ \frac{\partial \psi}{\partial p} = e^T \frac{\partial x}{\partial p} \]

\[ \frac{\partial f}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial f}{\partial p} = \frac{\partial y}{\partial p} \]

evaluated at \( x = x^{(o)} \) and \( p = p^{(o)} \)

In most cases \( \frac{\partial y}{\partial p} = 0 \)
\( \frac{\partial f}{\partial x} \equiv J \) has already been constructed and factorized into LU factors at \( x = x^{(o)} \) and \( p = p^{(o)} \)

\[
\frac{\partial \psi}{\partial p} = e^T \frac{\partial x}{\partial p} = e^T J^{-1} \left[ \frac{\partial y}{\partial p} - \frac{\partial f}{\partial p} \right]
\]

\[
= u^T \left[ \frac{\partial y}{\partial p} - \frac{\partial f}{\partial p} \right]
\]

\( u^T = e^T J^{-1} \), or \( [J]^T u = e \)

If \( \psi \) is a nonlinear function of \( x \) and \( p \):

\[
\psi = g(x, p)
\]

\[
\frac{\partial \psi}{\partial p} = \frac{\partial g}{\partial x} \frac{\partial x}{\partial p} + \frac{\partial g}{\partial p} = e_g^T \frac{\partial x}{\partial p} + \frac{\partial g}{\partial p}
\]

\[
\frac{\partial g}{\partial x} \equiv e_g^T
\]
Algorithm

1. Compute the solution of the resistive circuit at \( \mathbf{p} = \mathbf{p}^{(o)} \)
   
   \{save the \( \mathbf{LU} \) factors at the solution \( \mathbf{x}^{(o)} \)\}

2. Construct \( \mathbf{B} = \frac{\partial \mathbf{y}}{\partial \mathbf{p}} - \left[ \frac{\partial \mathbf{f}}{\partial \mathbf{p}} \right] \) and \( \frac{\partial \mathbf{g}}{\partial \mathbf{x}} \equiv \mathbf{e}_{g}^{T} \)

3. Solve the transpose (or adjoint) system \( \mathbf{U}^{T} \mathbf{L}^{T} \mathbf{u} = \mathbf{e}_{g} \)

4. Find \( \frac{\partial \psi}{\partial \mathbf{p}} = \mathbf{u}^{T} \mathbf{B} + \frac{\partial \mathbf{g}}{\partial \mathbf{p}} \)