Chapter Eight

INTEGRATION FORMULAS

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Transient Analysis

Linear and nonlinear resistive-inductive-capacitive-memristive (RLCM) circuits and systems
Differential and Algebraic-Differential Equations

\[ \dot{x} = f(x, t), \quad x(t_0) = x_0 \]

\[ f(x, \dot{x}, t) = 0, \quad x(t_0) = x_0 \]
Initial-Value Problem

• Given

\[ \dot{x} = f(x,t), \quad x(t_0) = x_0 \]

• Assume **unique solution** for every initial condition: No two trajectories intersect

![Tangent field](image)

**Figure 9.2: Tangent field** \[ \dot{x} = f(x, t) \]
**Theorem:** Given $\frac{dx}{dt} = f(x,t)$, if $f(x,t)$ satisfies Lipschitz conditions, then a unique solution exists in the interval $t \in [0, \infty)$ for any initial condition $x(0) = x_0$

**Lipschitz condition:**

$$||f(x,t) - f(y,t)|| \leq L(t)||x-y||$$

for all $t \in [0, \infty)$ and all $x, y \in \mathbb{R}^n$ where $L$ is a piecewise-continuous function in $t$.

- We’ll assume unique solutions for $\dot{x} = f(x, t)$ and $f(x, \dot{x}, t) = 0$ for any initial conditions.
Numerical Solution of $\dot{x} = f(x, t)$

Stable analytical solution and stable numerical solution
Numerical Solution of

\[ \dot{x} = f(x, t) \]

Stable analytical solution, stable numerical solution (-), and unstable numerical solution (*)
Numerical Solution of \( \dot{x} = f(x, t) \)

Unstable analytical solution, stable numerical solution (-), and unstable numerical solution (*)
Numerical Solution of \[ \dot{x} = f(x, t) \]

Taylor series expansion of \( x(t) \) at \( t_{n-1} \) (scalar case), assuming higher order derivatives exist; \( h = t_n - t_{n-1} \)

\[
x(t_n) = x(t_{n-1}) + h \frac{dx}{dt}(t_{n-1}) + \frac{1}{2!} h^2 \frac{d^2x}{dt^2}(t_{n-1}) + \frac{1}{3!} h^3 \frac{d^3x}{dt^3}(t_{n-1}) + \cdots
\]

\[
x(t_n) \simeq x(t_{n-1}) + h \frac{dx}{dt}(t_{n-1})
\]

Truncation Error:

\[
\frac{1}{2!} h^2 \frac{d^2x}{dt^2}(t_{n-1}) + \frac{1}{3!} h^3 \frac{d^3x}{dt^3}(t_{n-1}) + \cdots
\]
Numerical Solution of Forward Euler Formula (F.E.)

\[ \dot{x} = f(x, t) \]

\[ x_n = x_{n-1} + h\dot{x}_{n-1} \]

\[ x_n = x_{n-1} + hf(x_{n-1}, t_{n-1}) \]

Local Truncation Error:

\[ LTE \approx \frac{1}{2} h^2 \frac{d^2 x}{dt^2}(t_{n-1}) \]
Stability (and accuracy)

Test Equation: \[ \dot{x} = \lambda x, \quad x(0) = x_0, \]
a scalar linear differential equation with zero input.

Exact solution: \[ x(t) = x_0 e^{\lambda t}, \quad t \geq 0 \]

If \( \lambda < 0 \) (or if \( R_e\{\lambda \} < 0 \)), \( x(t) \to 0 \) as \( t \to \infty \) (stable)
If \( \lambda > 0 \) (or if \( R_e\{\lambda \} > 0 \)), \( x(t) \to \infty \) as \( t \to \infty \) (unstable)

Numerical solution using F.E.:

\[
x_n = x_{n-1} + h \dot{x}_{n-1} \\
= x_{n-1} + h \lambda x_{n-1} \\
= (1 + h\lambda) x_{n-1} \\
= (1 + h\lambda)^n x_0 \quad \text{Assuming constant } h
\]
• This means that if

\[ |1 + h\lambda| < 1, \ x_n \to 0 \text{ as } n \to \infty \]

and if \[ |1 + h\lambda| > 1, \ x_n \to \infty \text{ as } n \to \infty \]

\[ \therefore \text{ If } \lambda < 0, \text{ need } |1 + h\lambda| < 1 \Rightarrow |h\lambda| < 2 \text{ or } h < \frac{2}{|\lambda|} \]

Suppose \( x = -10^6x \), then \( h < \frac{2}{|10^6|} \)

Note: If \( \lambda > 0 \) => need \( |1 + \lambda| > 1 \), which is true for all \( h > 0 \)
Consider another test equation:

\[ \dot{x} = Ax \quad x(0) = x_0 \]

where A is a matrix of dimension greater than 1. Assume A to have distinct eigenvalues: \( A = T \Lambda T^{-1} \)

where \( \Lambda = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \ddots \\ & & & \lambda_m \end{bmatrix} \)
\( \lambda_i \) are the eigenvalues of \( \mathbf{A} \) (real or complex)

(i) If \( \Re\{\lambda_i\} < 0 \), for all \( i = 1, \ldots, n \) then the system is absolutely stable.

(ii) If \( \Re\{\lambda_i\} = 0 \) for some \( i \) (i.e., \( \lambda_i \) on imaginary axis) and simple and all other eigenvalues in left half plane (lhp), then system is oscillatory.

(iii) If there are multiple eigenvalues on imaginary axis and/or if at least one eigenvalue in right half plane (rhp), \( \Re\{\lambda_j\} > 0 \) for some \( j \), then system is unstable.
Forward Euler Formula

\[ x_n = x_{n-1} + h\dot{x}_{n-1} = x_{n-1} + hAx_{n-1} = (I + hA)x_{n-1} \]

\[ = (I + hT\Lambda T^{-1})x_{n-1} = T(I + h\Lambda)T^{-1}x_{n-1} \]

\[ = T(I + h\Lambda)T^{-1}T(I + h\Lambda)T^{-1}x_{n-2} \]

\[ = T(I + h\Lambda)^nT^{-1}x_0 \]

\[ x_n = T \begin{bmatrix} (1 + h\lambda_1)^n \\ (1 + h\lambda_2)^n \\ \vdots \\ (1 + h\lambda_m)^n \end{bmatrix} T^{-1}x_0 \]
If all $\Re\{\lambda_i\} < 0$, then $x(t) \to 0$ as $t \to \infty$ (absolutely stable)

In this case $x_n \to 0$ as $n \to \infty$ iff $|1 + h\lambda_i| < 1$ for all $i$

Let $\lambda_{\text{max}} = a + jb$, then $|1 + h\lambda_i| = |1 + ha + jhb| < 1$

$(1 + ha)^2 + (hb)^2 = 1$ represents a circle of radius 1 and centered at $(-1, 0)$:

If all $\Re\{\lambda_i\} < 0$, then $x(t) \to 0$ as $t \to \infty$ (absolutely stable)
If all $R_e\{\lambda_j\} < 0$, then $x(t) \to 0$ as $t \to \infty$ (absolutely stable)

Region of stability of F.E. in the $h\lambda,-$plane
F.E. Integration Formula

- Example: 2nd-order system: $\lambda = -1, -10^6$ (stiff system)
Forward Euler not suitable for stiff stable systems
Backward Euler Formula

- Taylor series expansion of $x(t)$ at $t_n$, assuming higher order derivatives exist.

\[
x(t_{n-1}) = x(t_n) - h \frac{dx}{dt}(t_n) + \frac{1}{2!} h^2 \frac{d^2x}{dt^2}(t_n) - \frac{1}{3!} h^3 \frac{d^3x}{dt^3}(t_n) + \cdots
\]

\[
x(t_{n-1}) \approx x(t_n) - h \frac{dx}{dt}(t_n)
\]

\[
x(t_n) \approx x(t_{n-1}) + h \frac{dx}{dt}(t_n)
\]

Truncation Error:

\[
LTE \approx \frac{1}{2} h^2 \frac{d^2x}{dt^2}(t_n)
\]
Backward Euler Formula (BE)

\[ x_n = x_{n-1} + h \dot{x}_n \] (Implicit Formula)

Apply to \( \dot{x} = f(x, t) \), \( x(0) = x_0 \)

get

\[ x_n = x_{n-1} + h f(x_n, t_n) \]

\[ x_n - hf(x_n, t_n) = x_{n-1} \]

\[ g(x_n) = x_{n-1} \]

Solve using Newton's Method.
Stability of BE formula

Consider test eqn.  \( x = \lambda x \), \( x(0) = x_0 \)

\[
\begin{align*}
x_n &= x_{n-1} + h x_n = x_{n-1} + h \lambda x_n \\
(1 - h \lambda) x_n &= x_{n-1} \\
x_n &= \frac{1}{(1 - h \lambda)} x_{n-1} = \frac{1}{(1 - h \lambda)^n} x_0
\end{align*}
\]

If \( \lambda < 0 \) \( \Rightarrow x(t) = x_0 e^{\lambda t} \) and \( x(t) \rightarrow 0 \) as \( t \rightarrow \infty \)
(Absolutely stable)
Numerical Solution

\[ x_n \to 0 \text{ as } n \to \infty \text{ iff } \frac{1}{|1 - h\lambda|} < 1 \]

or \[ |1 - h\lambda| > 1 \]

which is true for all \( h > 0 \)
Consider \( \dot{x} = Ax, \ x(0) = x_0 \)

\[
A = T \Lambda T^{-1}
\]

\[
\Lambda = \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_m
\end{bmatrix}
\]

\[
(B.E.): \quad x_n = x_{n-1} + hAx_n
\]

\[
(I - hA) x_n = x_{n-1}
\]

\[
x_n = (I-hA)^{-1} x_{n-1}
\]

\[
x_n = [T(I - h \Lambda) T^{-1}]^{-1} x_{n-1}
\]

\[
x_n = T(I - h \Lambda)^{-1} x_0
\]
If $R_e \{ \lambda_i \} < 0$ for all $i$, then the system is absolutely stable and numerical solution generated by B. E. is absolutely stable for $h > 0$.

If $R_e \{ \lambda_i \} > 0$ for some $i$, the system is unstable and $x(t) \to \infty$ as $t \to \infty$.

For numerical solution $x_n \to \infty$ as $n \to \infty$.

\[ \lambda_1 > 1 \text{ for at least one } \lambda_i, \text{ or } |1 - h \lambda_i| < 1 \]
Region of stability of B.E. in $h\lambda$-plane

$|1 - h \lambda_i| = |1 - ha - jhb| < 1$ is the region of instability. Boundary of the region is

$$(1-ha)^2 + (hb)^2 = 1$$

which is a circle centered at (+1,0) in the complex $h\lambda$-plane with radius 1.
Region of stability of B.E. in $h\lambda$-plane.

Stable for all $h > 0$ when all $\lambda_i$ are in lhp (absolutely stable system)
Unstable when $h < 2/\lambda_{\text{min}}$
Remark: For an increasing transient (with decreasing slope) F.E. overshoots the actual solution, B.E. undershoots: Averaging Formula: Trapezoidal Rule
Trapezoidal Rule (T.R.)

\[ x_n = x_{n-1} + \frac{h}{2} (x_{n-1} + x_n), \text{ Implicit, one-step formula} \]

\[ \dot{x} = f(x, t) \]

\[ x_n = x_{n-1} + \frac{h}{2} \left( f(x_{n-1}, t_{n-1}) + f(x_n, t_n) \right) \]

\[ x_n - \frac{h}{2} f(x_n, t_n) = x_{n-1} + \frac{h}{2} f(x_{n-1}, t_{n-1}) \]

\[ g(x_n) = y \text{ known} \]

Use Newton's Method to find \( x_n \)
To study stability properties of T.R. apply to a test example: \( \dot{x} = \lambda x \), \( x(0) = x_0 \)

\[
x_n = x_{n-1} + \frac{h}{2} \left( \lambda x_{n-1} + \lambda x_n \right)
\]

\[
\left(1 - \frac{h}{2} \lambda \right)x_n = x_{n-1} + \frac{h}{2} \lambda x_{n-1}
\]

\[
\left(1 - \frac{h\lambda}{2} \right)x_n = \left(1 + \frac{h\lambda}{2} \right)x_{n-1}
\]

\[
x_n = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} x_{n-1}
\]

\[
= \frac{\left(1 + \frac{h\lambda}{2} \right)^n}{\left(1 - \frac{h\lambda}{2} \right)^n} x_0
\]
If $R_e\{\lambda\} < 0 \Rightarrow$ Stable System

We want

$$\left| \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right| < 1$$

Let $\lambda = a + jb, \ a < 0$

$$\frac{(1 - \frac{h|a|}{2})^2 + (\frac{hb}{2})^2}{(1 + \frac{h|a|}{2})^2 + (\frac{hb}{2})^2} < 1$$

True for all $h > 0$
If $\Re\{\lambda\} > 0 \Rightarrow \text{Unstable}$

Want \[
\frac{|1 + h\lambda/2|}{|1 - h\lambda/2|} > 1
\]

$\lambda = a + jb, a > 0$

\[
\frac{(1 + \frac{ha}{2})^2 + (\frac{hb}{2})^2}{(1 - \frac{ha}{2})^2 + (\frac{hb}{2})^2} > 1
\]

True for all $h > 0$
Region of Stability of TR

Stability of numerical solution obtained by T.R. matches the stability of the original system
Case of an Oscillator

\[ C \frac{dv_C}{dt} = -i_L \]

\[ L \frac{di_L}{dt} = v_C \]

\[ \begin{bmatrix} \dot{v}_C \\ \dot{i}_L \end{bmatrix} = \begin{bmatrix} 0 & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C \\ i_L \end{bmatrix} \]
Eigenvalues:

\[
\det \begin{bmatrix}
\lambda & \frac{1}{C} \\
-\frac{1}{L} & \lambda \\
\end{bmatrix} \Rightarrow \lambda^2 - \frac{1}{LC} = 0
\]

\[
\lambda = \sqrt{-\frac{1}{LC}} = \pm j \frac{1}{\sqrt{LC}}
\]

freq = \frac{1}{\sqrt{LC}}

\lambda\text{-plane}

\star \quad \star

j \frac{1}{\sqrt{LC}}

\star \quad \star

-j \frac{1}{\sqrt{LC}}
F.E. 
\[ x_n = (1 + h\lambda)^n x_0 \]
\[
\left| 1 + jh \frac{1}{\sqrt{LC}} \right| = \left( 1 + \frac{h^2}{LC} \right)^{1/2} > 1 \forall h
\]
Unstable, \( x_n \to \infty \) as \( n \to \infty \)

B.E. 
\[ x_n = \left| \frac{1}{(1-h\lambda)^n} \right| x_0 \]
\[
\left| 1 - j\frac{h}{\sqrt{LC}} \right| = \frac{1}{\left( 1 + \frac{h^2}{LC} \right)^{1/2}} > 1 \quad \forall h > 0 \Rightarrow \text{stable}
\]
Stable, \( x_n \to 0 \) as \( n \to \infty \)

T.R. 
\[ x_n = \left( \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \right)^n x_0 \]
\[
\left| \frac{1 + j\frac{h}{2\sqrt{LC}}}{1 - j\frac{h}{2\sqrt{LC}}} \right| = \left( \frac{1 + \frac{h^2}{4LC}}{1 + \frac{h^2}{4LC}} \right) = 1
\]
T.R Oscillates for any \( h > 0 \) same as the actual solution
One drawback of T.R.

For stiff systems, some $|\lambda|$'s are small (slow transients) and some $|\lambda|$'s are large (fast transients).

If $h\lambda$ becomes very large $h|\lambda| \rightarrow \infty$

\[ x_n = \left( \frac{1 + h\lambda/2}{1 - h\lambda/2} \right)^n x_0 \]

\[ \left( \frac{1 + h\lambda/2}{1 - h\lambda/2} \right)^n \xrightarrow{h\lambda \rightarrow \infty} (-1)^n \]
Solution oscillates (numerically) around $x_0$, i.e.,

$$x_n \to (-1)^n x_0$$

instead of going to zero.

In contrast, using B.E.

$$x_n = \frac{1}{(1-h\lambda)^n} x_0$$

as $h|\lambda| \to \infty$, $x_n \to 0$
There are many different methods for solving ODEs:

- Taylor Series
- Runge-Kutta
- Extrapolation
- Multi-Value
- Multi-step methods
Linear Multistep Formulas

Consider \( \dot{x} = f(x,t), \ x(0) = x_0 \)

\[
x_n = \sum_{i=1}^{k} \alpha_i x_{n-i} + h \sum_{i=0}^{k} \beta_i \dot{x}_{n-i}
\]

If \( \beta_0 = 0 \) => explicit, \( \beta_0 \neq 0 \) => implicit
Linear Multistep Formulas

- A linear multistep formula (lmtf) approximates the solution $x(t)$ locally in the interval $t_{n-k} < t < t_n$ by an interpolating polynomial

$$p(t) = a_p t^p + a_{p-1} t^{p-1} + \ldots + a_0$$

of order $p \leq 2k$ that satisfy the points $x_{n-i}$ and its time derivatives $f_{n-i}$, $i=0,1,\ldots,k$.

- The formula should satisfy the basis functions

$$\{1, t, t^2, t^3, \ldots, t^p\}$$

because of linearity.
Another form:

\[
\sum_{i=0}^{k} (\alpha_i x_{n-i} + h \beta_i \dot{x}_{n-i}) = 0, \; \alpha_0 = -1, \; k = \# \text{ of steps}
\]
How to select $\alpha_i$ and $\beta_i$ for a k-th step \textit{lmf}

1. Select a number $p \leq 2k$

2. Select a set of polynomial basis functions \{1, t, t^2, ..., t^p\}

3. Compute the coefficients of the \textit{lmf} such that the formula is exact for each of the basis functions. The formula will then be exact for any polynomial of order $p$ because of linearity.

If $p < 2k$, some of the coefficients of the \textit{lmf} can be assigned arbitrary values.

\[
\sum_{i=0}^{k} (\alpha_i \cdot x_{n-i} + h \beta_i \cdot x_{n-i}) = 0, \quad \alpha_0 = -1
\]
Example

$k = 1$ (one-step formula)

$$\sum_{i=0}^{1} (\alpha_i x_{n-i} + h \beta_i \dot{x}_{n-i}) = 0, \quad \alpha_0 = -1$$

$$-x_n + \alpha_1 x_{n-1} + h (\beta_0 \dot{x}_n + \beta_1 \dot{x}_{n-1}) = 0$$

Order $p$ can be 1 or 2
• Choose \( p = 1 \)
• Basis functions: \( \{1, t\} \)

\[
-x_n + \alpha_1 x_{n-1} + h (\beta_0 \dot{x}_n + \beta_1 \dot{x}_{n-1}) = 0
\]

• 1: \(-1 + \alpha_1 = 0 \Rightarrow \alpha_1 = 1\)
• t: \(-t_n + \alpha_1 t_{n-1} + h(\beta_0 + \beta_1) = 0\)
• Let \( t_{n-1} = 0\), then \( t_n = h\)
\[
\begin{align*}
\alpha_1 &= 1 \\
-1 + \beta_0 + \beta_1 &= 0
\end{align*}
\]

- Choose \( \beta_0 = 0 \Rightarrow \beta_1 = 1 \Rightarrow \text{explicit} \\
\begin{align*}
-x_n + x_{n-1} + h x_{n-1} &= 0 \quad \text{(F.E.)}
\end{align*}

- Choose \( \beta_1 = 0 \Rightarrow \beta_0 = 1 \Rightarrow \text{implicit} \\
\begin{align*}
-x_n + x_{n-1} + h x_n &= 0 \Rightarrow \quad \text{(B.E.)}
\end{align*}

- Choose \( \beta_0 = \mu \) , \( \beta_1 = 1 - \mu \) \\
\begin{align*}
-x_n + x_{n-1} + h \left( \mu x_n + (1 - \mu) x_{n-1} \right) &= 0,
\end{align*}

one-step 1st order formula (if \( \mu \neq 1/2 \)) \\
(If \( \mu = \frac{1}{2} \Rightarrow \text{T.R.} \))
• Choose $p = 2$

$$-x_n + \alpha_1 x_{n-1} + h (\beta_0 \dot{x}_n + \beta_1 \dot{x}_{n-1}) = 0$$

Basis functions: $\{1, t, t^2\}$

1: $-1 + \alpha_1 = 0 \Rightarrow \alpha_1 = 1$

t: $-t_n + \alpha_1 t_{n-1} + h (\beta_0 + \beta_1) = 0$

t$^2$: $-t_n^2 + \alpha_1 t_{n-1}^2 + h (\beta_0 2t_{n-1} + \beta_1 2t_n) = 0$
1. $\alpha_1 = l$

2. $-h + 0 + h(\beta_0 + \beta_1) = 0$
   \[\beta_0 + \beta_1 = l\]

3. $-h^2 + \alpha_0 + h(\beta_0 0 + \beta_1 2h) = 0$
   \[-h^2 + 2\beta_1 h^2 = 0 \Rightarrow -l + 2\beta_1 = 0\]
   \[\beta_1 = l/2, \quad \beta_0 = l/2, \quad \alpha_1 = 1\]

\[-x_n + x_{n-1} + h \left( \frac{1}{2} x_n + \frac{1}{2} x_{n-1} \right) = 0 \text{ (Trapezoidal Rule order 2)}\]
Suppose $k = 2$ (2-step formula)

\[ \sum_{i=0}^{2} \left( \alpha_i x_{n-i} + h \beta_i \dot{x}_{n-i} \right) = 0 \]

\[-x_n + \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + h (\beta_0 \dot{x}_n + \beta_1 \dot{x}_{n-1} + \beta_2 \dot{x}_{n-2}) = 0\]
p ≤ 4, since k = 2

Basis function  \{1, t, t^2, t^3, t^4\}

If p = 4, basis functions will generate five equations in five unknowns: \(\alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2\), which are functions of \(h_1\) and \(h_2\). When \(h_1 = h_2\) (constant step size) the coefficients become constants.

If p < 4, some coefficients \((\alpha_1, \alpha_2, \beta_0, \beta_1, \beta_2)\) are chosen arbitrarily (usually zero)
Suppose $p = 3$ ($k = 2$)

Basis functions $\{1, t, t^2, t^3\}$

\[-x_n + \alpha_1 x_{n-1} + \alpha_2 x_{n-2} + h_1 (\beta_0 \dot{x}_n + \beta_1 \dot{x}_{n-1} + \beta_2 \dot{x}_{n-2}) = 0\]

1: $-1 + \alpha_1 + \alpha_2 = 0$

t: $-(h_1 + h_2) + \alpha_1(h_2) + 0 + h_1(\beta_0 + \beta_1 + \beta_2) = 0$

t$^2$: $-(h_1 + h_2)^2 + \alpha_1(h_2)^2 + 0 + h_1(2\beta_0(h_1 + h_2) + 2\beta_1 h_2) = 0$

t$^3$: $-(h_1 + h_2)^3 + \alpha_1(h_2)^3 + 0 + h_1(3\beta_0(h_1 + h_2)^2 + 3\beta_1 h_2^2) = 0$

Four equations in five unknowns.
Suppose $\alpha_1 = 0$ (p=3, k=2)

\[ x_n = \alpha_2 x_{n-2} + h(\beta_0 \dot{x}_n + \beta_1 \dot{x}_{n-1} + \beta_2 \dot{x}_{n-2}) \]

Consider constant stepsize $h$. Put $t_{n-2} = 0$, then $t_{n-1} = h$, $t_n = 2h$

Basis functions $\{1, t, t^2, t^3\}$

\[
\begin{align*}
1 & : 1 = \alpha_2 \\
 t & : 2 = \beta_0 + \beta_1 + \beta_2 \\
 t^2 & : 4 = 4\beta_0 + 2\beta_1 \\
 t^3 & : 8 = 12\beta_0 + 3\beta_1 
\end{align*}
\]

Solution: $\alpha_2 = 1$, $\beta_0 = 1/3$, $\beta_1 = 4/3$, $\beta_2 = 1/3$, 

\[ x_n = x_{n-2} + \frac{h}{3}(\dot{x}_n + 4\dot{x}_{n-1} + \dot{x}_{n-2}) \]

Simpson’s Rule
Accuracy (errors)

Two sources of errors

• Roundoff errors: due to finite precision of floating-point arithmetic

• Truncation (or discretization) error due to method used (remains even with infinite precision)
Truncation Error

**Global Truncation Error (GTE):** difference between the computed solution and the true solution determined by the initial condition at $t_0$ (difficult to measure).

**Local Truncation Error (LTE):**
Error committed in one step in the numerical solution process. LTE is used to control the time-step $h$ (for accuracy).
actual solution starting at $x_0$  

$|x(t_n) - x_n| \Rightarrow \text{GTE at } n$  

actual solution starting at $x_{n-1}$
Local Truncation Error

• A linear \textit{k-step} formula of order \( p \) is derived on the assumption that the solution can be represented as a polynomial in \( t \) of order \( p \) that matches the values \( x_{n-i} \) \textit{and} its derivatives \( f_{n-i}, \ i = 1, \ldots, k \).

• Define

\[
L[x(t); h] = \sum_{i=0}^{k} [\alpha_i x(t_{n-i}) + h\beta_i \dot{x}(t_{n-i})]
\]
Local Truncation Error

- $L(1; h) = 0$
- $L(t; h) = 0$
- $L(t^p; h) = 0$
- $L(t^{p+1}; h) \neq 0$
Local Truncation Error

Expanding $x(t_{n-i})$ and $\dot{x}(t_{n-i})$ in Taylor series expansion about $t_n$

$$x(t_{n-i}) = \sum_{j=0}^{\infty} \frac{1}{j!} (t_{n-i} - t_n)^j x^{(j)}(t_n)$$

$$x^{(1)}(t_{n-i}) = \sum_{j=0}^{\infty} \frac{1}{j!} (t_{n-i} - t_n)^j x^{(j+1)}(t_n)$$

Put into

$$L[x(t); h] = \sum_{i=0}^{k} [\alpha_i x(t_{n-i}) + h \beta_i \dot{x}(t_{n-i})]$$
Local Truncation Error

Collect terms:

\[ L[x(t); h] = C_0 x(t) + C_1 h x^{(1)}(t) + \cdots + C_j h^j x^{(j)}(t) + \cdots \]

\[ C_0 = \alpha_0 + \alpha_1 + \alpha_2 + \cdots + \alpha_k \]

\[ C_1 = \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k - (\beta_0 + \beta_1 + \beta_2 + \cdots + \beta_k) \]

\[ C_j = \frac{1}{j!} (\alpha_1 + 2^j \alpha_2 + \cdots + k^j \alpha_k) - \frac{1}{(j-1)!} (\beta_1 + 2^{j-1} \beta_2 + \cdots + k^{j-1} \beta_k) \]
Local Truncation Error

- \( L(1; h) = C_0 = 0 \)
- \( L(t; h) = C_1 h = 0 \)
- \( L(t^p; h) = p!C_p h^p = 0 \)
- \( L(t^{p+1}; h) = (p+1)!C_{p+1} h^{p+1} \neq 0 \)
Local Truncation Error

From Taylor series expansion:

\[ L[x(t_n); h] = C_{p+1} h^{p+1} x^{(p+1)}(t_n) + O(h^{p+2}) = O(h^{p+1}) \]

\[ \text{LTE} = C_{p+1} h^{p+1} x^{(p+1)}(t_n) \]

\( C_{p+1} \) is the Error Constant
Local Truncation Error

• Also: \( L(t^{p+1};h) = (p+1)!C_{p+1}h^{p+1} \)

• \( C_{p+1}h^{p+1} = \left[1/(p+1)!!\right] L(t^{p+1};h) \)

• LTE = \( C_{p+1}h^{p+1} x^{(p+1)}(t_n) \) (from Taylor series)

\[= \left[1/(p+1)!!\right] L(t^{p+1};h) x^{(p+1)}(t_n) \) (from putting \( t^{p+1} \) in formula)\]
Example: LTE of F.E.

\[
\text{F.E.} : \quad -x_n + x_{n-1} + h\dot{x}_{n-1} = 0
\]

\[
x(t_{n-1}) = x(t_n) - h\dot{x}(t_n) + \frac{h^2}{2} \ddot{x}(t_n) - \frac{h^3}{6} \dddot{x}(t_n) + \ldots
\]

\[
\dot{x}(t_{n-1}) = \dot{x}(t_n) - h\ddot{x}(t_n) + \frac{h^2}{2} \dddot{x}(t_n) - \frac{h^3}{6} \ddddot{x}(t_n) + \ldots
\]

\[
\text{LTE} = \quad -x(t_n) + (x(t_n) - h\dot{x}(t_n) + \frac{h^2}{2} \ddot{x}(t_n) + \ldots)
\]

\[
+ h\dot{x}(t_n) - h^2\ddot{x}(t_n) + \frac{h^3}{2} \dddot{x}(t_n) - \ldots)
\]

\[
= -\frac{h^2}{2} \dddot{x}(t_n)
\]
LTE of F.E.

Second method

• Put $t^2$ in formula:

$$L[t^2; h] = -t_n^2 + t_{n-1}^2 + 2ht_{n-1}$$

If $t_{n-1} = 0$, then $t_n = h$, and $L[t^2; h] = -h^2$.

$$LTE = -\frac{1}{2}h^2x^{(2)}(t_n)$$
LTE of B.E.

\[ B.E.: \quad -x_n + x_{n-1} + h\dot{x}_n = 0 \]

1- Taylor Series:

\[ LTE = -x(t_n) + (x(t_n) - h\dot{x}(t_n) + \frac{h^2}{2} \ddot{x}(t_n) - \ldots) \]

\[ + h\dot{x}(t_n) = \frac{h^2}{2} \ddot{x}(t_n) \]
LTE of B.E.

\[ \text{B.E.: } -x_n + x_{n-1} + h\dot{x}_n = 0 \]

2- Put \( t^2 \) in formula:

\[ L[t^2; h] = -t_n^2 + t_{n-1}^2 + 2ht_n \]

If \( t_{n-1} = 0 \), then \( t_n = h \), and \( L[t^2; h] = h^2 \).

\[ \text{LTE} = \frac{1}{2}h^2x^{(2)}(t_n) \]
LTE of T.R.

\[ T.R.: \quad -x_n + x_{n-1} + \frac{h}{2} (\dot{x}_n + \dot{x}_{n-1}) = 0 \]

\[ |LTE| = -x(t_n) + (x(t_n) - h\dot{x}(t_n) + \frac{h^2}{2} \ddot{x}(t_n)) \]

\[ -\frac{h^3}{6} \dddot{x}(t_n) + O(h^4) \cdots + \frac{h}{2} \dot{x}(t_n) + \frac{h}{2} (\dot{x}(t_n) - h\ddot{x}(t_n) + \frac{h^2}{2} \dddot{x}(t_n) + O(h^3)) \]

\[ = -\frac{h^3}{6} \dddot{x}(t_n) + \frac{h^3}{4} \dddot{x}(t_n) + O(h^4) \]

\[ = \frac{1}{12} h^3 \dddot{x}(t_n) \]
LTE of T.R.

T.R.: \(-x_n + x_{n-1} + \frac{h}{2}(\dot{x}_n + \dot{x}_{n-1}) = 0\)

2- Put \(t^3\) in formula:

\[
L[t^3; h] = -t_n^3 + t_{n-1}^3 + \frac{3}{2} h (t_n^2 + t_{n-1}^2)
\]

Let \(t_{n-1} = 0\), then \(t_n = h\), and \(L[t^3; h] = \frac{1}{2} h^3\).

\[
LTE = \frac{1}{3!} \frac{1}{2} h^3 x^{(3)}(t_n) = \frac{1}{12} h^3 x^{(3)}(t_n)
\]
How to estimate LTE in practice

Use **Finite Difference Interpolation**:

\[
\frac{dx_n}{dt} = \frac{x_n - x_{n-1}}{t_n - t_{n-1}} \equiv x[t_n, t_{n-1}]
\]

\[
\frac{d^2x_n}{dt^2} \approx 2 \frac{x[t_n, t_{n-1}] - x[t_{n-1}, t_{n-2}]}{t_n - t_{n-2}}
\]

\[
\frac{x_n - x_{n-1}}{t_n - t_{n-1}} - \frac{x_{n-1} - x_{n-2}}{t_{n-1} - t_{n-2}}
\]

\[
\approx 2 \frac{x_n - x_{n-1}}{t_n - t_{n-1}} - \frac{x_{n-1} - x_{n-2}}{t_{n-1} - t_{n-2}} - \frac{x_{n-2} - x_{n-3}}{t_{n-2} - t_{n-3}}
\]

\[
\frac{d^kx_n}{dt^k} = k! \frac{x[t_{n1}t_{n-1}, \ldots t_{n-k+1}] - x[t_{n-1}, \ldots t_{n-k}]}{t_n - t_{n-k}}
\]
Note: LTE decreases with h, but when h decreases, round-off error increases.
Linear Multistep Formulas

\[ \sum_{i=0}^{k} (\alpha_i x_{n-i} + h\beta_i x_{n-i}) = 0 \]

- \( k \) = # of steps
- \( \beta_0 = 0 \) => explicit formula
- \( \beta_0 \neq 0 \) => implicit formula
- \( p = \text{order} = \text{highest order of basis functions} \{l, t, ..., t^p\} \) that satisfy the formula

\[ \text{LTE} = C_{p+1} h^{p+1} \frac{d^{p+1}x}{dt^{p+1}} = O(h^{p+1}) \]

(order can be also found by computing LTE expression)

- \( p \leq 2k \). If \( p < 2k \), some coefficients are chosen arbitrarily (usually zero, but not necessarily)
Examples

What is the order of each of the following formulas?

(a) $x_n = x_{n-1} + x_{n-2} + h(x_n + x_{n-1})$

1: $1 = 1 + 1 \times$ (Does not qualify)

(b) $x_n = \frac{1}{2} x_{n-1} + \frac{1}{2} x_{n-2} + h(x_n + x_{n-1})$

1: $1 = \frac{1}{2} + \frac{1}{2} \checkmark$

$t$: $t_n = \frac{1}{2} t_{n-1} + \frac{1}{2} t_{n-2} + h(1 + 1)$

$2h = \frac{1}{2} h + 0 + 2h$

Order = 0
(c) $x_n = \frac{1}{2} x_{n-1} + \frac{1}{2} x_{n-2} + h \left( \dot{x}_n + \frac{1}{2} \dot{x}_{n-1} \right)$

1: \[ 1 = \frac{1}{2} + \frac{1}{2} \quad \checkmark \]

t: \[ t_n = \frac{1}{2} t_{n-1} + \frac{1}{2} t_{n-2} + h \left( 1 + \frac{1}{2} \right) \]

\[ 2h = \frac{1}{2} h + 0 + \frac{3}{2} h \quad \checkmark \]

$ t^2$: \[ (2h)^2 = \frac{1}{2} (h)^2 + 0 + h \left( 2t_n + \frac{1}{2} 2t_{n-1} \right) \]

\[ 4h^2 = \frac{1}{2} h^2 + 5h^2 \quad \times \]

Order = 1
STABILITY PROPERTIES OF LINEAR MULTISTEP FORMULAS (LMF)
Stability

Consider lmf

\[
\sum_{i=0}^{k} (\alpha_i x_{n-i} + h \beta_i \dot{x}_{n-i}) = 0
\]

Apply formula to test equation \( x = \lambda x \) to get:

\[
\sum_{i=0}^{k} \alpha_i x_{n-i} + h \lambda \sum_{i=0}^{k} \beta_i x_{n-i} = 0
\]

\[
\sum_{i=0}^{k} (\alpha_i + h \lambda \beta_i) x_{n-i} = 0
\]

\[
\gamma_0 x_n + \gamma_1 x_{n-1} + \ldots + \gamma_n x_{n-k} = 0 \quad \text{(difference equation)}
\]

Coefficients \( \gamma_i \)'s are dependent on \( h \lambda \):
The **Characteristic polynomial** of the difference equation is

\[ P(z) = \gamma_0 z^k + \gamma_1 z^{k-1} + \ldots + \gamma_k = 0 \]

Assume \( P(z) \) has \( k \) distinct roots, then the solution of difference equation is

\[ x_n = \sum_{j=1}^{k} c_j z_j^n \]

- If \( \text{Re}\{\lambda\} < 0 \rightarrow x(t) \rightarrow 0 \text{ as } t \rightarrow \infty \) (absolutely stable)

- To have \( x_n \rightarrow 0 \text{ as } n \rightarrow \infty \) requires
  \[ |z_i| < 1 \quad \text{for all } i = 1, \ldots, k \]
The characteristic polynomial can be written as

\[ P(z, h\lambda) = \sum_{i=0}^{k} \alpha_i z^{k-i} + h\lambda \sum_{i=0}^{k} \beta_i z^{k-i} = 0 \]

\[ = \rho(z) + h\lambda \sigma(z) = 0 \]

- To study the stability of Im f assume Re\{ \lambda\} < 0, (stable test equation) and investigate the location of the roots of \( P(z, h\lambda) = 0 \) as a function of \( h\lambda \)
- For numerical stability, the roots of \( P(z, h\lambda) \) should be less or equal to 1 (with those equal to 1 being simple)
- The stability properties of the formula are studied by analyzing the root locii of \( P(z, h\lambda) = 0 \) as \( |h\lambda| \) varies from 0 to \( \infty \).
- First, the formula should behave 'well' when \( h \to 0 \).
(i) Convergence

\[
\lim_{h \to 0, \ nh = b-a} x_n = x(t_n) \quad \text{for all } t \in [a, b]
\]

Convergence implies that the global truncation Error → 0 when \( h \to 0, \ n \to \infty \) and \( nh = \text{constant} \),
Practical Check of Convergence

- Can also check rate of convergence
(ii) Consistency

A \textit{Imf} is consistent if its order is \( p \geq 1 \); that is, at least it satisfies \( \{1, t\} \)

Consider

\[
P(z, h\lambda) = \sum_{i=0}^{k} \alpha_i z^{k-i} + h\lambda \sum_{i=0}^{k} \beta_i z^{k-i} = 0
\]

\[
= \rho(z) + h\lambda \sigma(z)
\]
Put basis \{1\} in lmf:

\[ \sum_{i=0}^{k} \alpha_i = 0 \quad \alpha_0 = -1 \]

\[ \rho(1) = \sum_{i=0}^{k} \alpha_i z^{k-i} \bigg|_{z=1} = \sum_{i=0}^{k} \alpha_i = 0 \]

\[ \therefore 1 \text{ is a root of } P(z, h\lambda)|_{h\lambda=0} = \rho(z) = 0 \]

If a consistent lmf is applied to solve \( \dot{x} = 0 \), \( x(0) = c \) (i.e., \( \lambda = 0 \)) whose solution is \( x(t) = c \), \( t \geq 0 \), any initial numerical error should die out if the method is to be stable. The numerical solution is given by

\[ x_n = \sum_{i=1}^{k} \alpha_i x_{n-i} \]
The solution sequence can be expressed in closed form as

\[ x_n = \sum_{i=1}^{k} c_i z_i^n \]

where \( z_i \) are the roots of \( \rho(z) = 0 \).

From consistency, \( z_1 = 1 \) is a root of \( \rho(z) = 0 \).

This root is called the **principal** root.

To ensure stability, the roots of \( \rho(z) = 0 \), \( |z_i| \leq 1 \), \( i = 1, 2, ..., k \) and any root of modulus 1 is simple.

(When \( \dot{x} = \lambda x \), \( \lambda \neq 0 \) and \( h\lambda \neq 0 \), the principal root “tracks” the actual solution.)
(iii) Zero-Stability

If all the roots of $\rho(z) = 0$ satisfy the condition $|z_i| \leq 1$, $i = 1, 2, \ldots, k$, and any root of modulus 1 is simple, then the lmf is zero-stable.

• **Theorem** If a lmf is consistent and zero-stable, then it is **convergent**.

• **Theorem** No zero-stable linear k-step formula can have order exceeding $k + 1$ when $k$ is odd, or exceeding $k + 2$ when $k$ is even.
Examples

Adams Formulas: \(-x_n + \alpha_1 x_{n-1} + h \sum_{i=0}^{k} \beta_i \dot{x}_{n-1} = 0\)

\[-1 + \alpha_1 = 0 \implies \alpha_1 = 1\]

If \(k=1\),

\[\beta_0 = 0, \beta_1 \neq 0 = 1 \implies \text{F.E.}\]
\[\beta_0 \neq 0 = 1, \beta_1 = 0 \implies \beta_0 \neq 0 = 1/2, \beta_1 \neq 0 = 1/2 \implies \text{T.R.}\]

\[\rho(z) = -z^k + z^{k-1} = 0\]

Roots at 1 and 0

Formulas are zero-stable if coefficients \(\beta_i\) are chosen so that formulas satisfies basic function \(\{t\}\), in addition to \(\{1\}\).
k=1    =>    Backward Euler (B.E.)

\[ \sum_{i=0}^{k} \alpha_i x_{n-i} + h\beta_0 \dot{x}_n = 0 \]

\[ \rho(z) = -z^2 + \frac{4}{3} z - \frac{1}{3} = 0 \]

Roots are 1 and 1/3    =>    Zero-Stable

Satisfies \{1, t, t^2\}    =>    Consistent    =>    Convergent
k = 2, p = 3 => consistent

\[ \rho(z) = -z^2 - 4z + 5 = 0 \]

Roots are 1, -5 => not zero-stable, therefore not convergent
Apply to test equation:

\[ \dot{x} = \lambda x \]

get difference eqn.: \[ \sum_{i=0}^{k} (\alpha_i + h\lambda \beta_i) x_{n-i} = 0 \]

whose solution is: \[ x_n = \sum_{i=1}^{k} c_i z_i^n \]
If $h\lambda = 0$, $P(z; h\lambda) = \rho(z) = 0$ where $z_i$ are the roots of characteristics polynomial:

\[
P(z; h\lambda) = \sum_{i=0}^{k} \alpha_i z^{k-i} + h\lambda \sum_{i=0}^{k} \beta_i z^{k-i} = 0
\]

\[
\equiv \rho(z) + h\lambda \sigma(z)
\]

If $h\lambda = 0$, $P(z; h\lambda) = \rho(z) = 0$  

$z_1 = 1$ is a root of $\rho(z) = 0$ (principal root)
Zero-Stable

If all roots of $\rho(z) = 0$ are within or on unit circle in complex plane, with any roots on the circle being simple.

Consistency

Order of $Im f \rho \geq 1$
Convergent

Consistent + zero-stable

Region of Stability

Is defined as a region $S$ in the complex $h\lambda$-plane where the zeros $z_i$ of the "stability" polynomial $P(z; h\lambda) = 0$ fulfill $|z_i| < 1$, $i = 1, 2, \ldots, k$. 
The boundary of the stability region is found from
\[ P(z; h\lambda) = \rho(z) + h\lambda\sigma(z) = 0 \]

\[ h\lambda = -\frac{\rho(z)}{\sigma(z)} ; |z| = 1 \]

\[ = -\frac{\rho(e^{i\theta})}{\sigma(e^{i\theta})}, z = e^{i\theta} \quad 0 \leq 0 \leq 2\pi \]
Examples

B.E.: $-x_n + x_{n-1} + h \dot{x}_n = 0 \implies \dot{x} = \lambda x$

$P(z; h\lambda) = -z + 1 + h\lambda z = 0$

\[
\begin{array}{c|c}
  z & h\lambda \\
  \hline
  1 & 0 \\
  -1 & 2 \\
  j & 1+j \\
  0 & \infty \\
\end{array}
\]

z-plane

h\lambda-plane

Stable

unstable
When \( h \lambda \neq 0 \), it can be shown that \( z_1(h \lambda) \) is a continuous function of \( h \lambda \). The other roots \( z_i \), \( i = 2, \ldots, k \) are called extraneous roots.

The numerical solution

\[
x_n = \sum_{i=1}^{k} c_i z_i^n
\]

will be a good approximation to the exact solution if

\( |z_i| < |z_1|, \ i = 2, \ldots, k \).
**Absolute Stability**

A *lmf* is said to be absolutely stable for a given $h \lambda$ if the roots $z_i$ of $P(z; h \lambda) = 0$ satisfy $|z_i| < 1$ for $i = 1, 2, ..., k$.

**Relative Stability**

A *lmf* is said to be relatively stable for a given $h \lambda$ if the roots of $P(z; h \lambda) = 0$ satisfy $|z_i| < |z_1|$ for $i = 2, ..., k$. (Note in this case $|z_1|$ could be < 1 (stable) or >1 (unstable).)
Example: 2nd-order BDF with constant h

\[
-x_n + \frac{4}{3} x_{n-1} - \frac{1}{3} x_{n-2} + \frac{2}{3} h \dot{x}_n = 0
\]

\[P(z; h\lambda) = -z^2 + \frac{4}{3} z - \frac{1}{3} + \frac{2}{3} h \lambda z^2 = 0\]

\[
\left(-1 + \frac{2}{3} h\lambda\right) z^2 + \frac{4}{3} z - \frac{1}{3} = 0
\]

\[
z_1 = \frac{-2 - \sqrt{4 - 3 + 2h\lambda}}{-3 + 2h\lambda}, \quad z_2 = \frac{-2 + \sqrt{1 + 2h\lambda}}{-3 + 2h\lambda}
\]

It can be shown that \(|z_1| < 1, |z_2| < 1\) for \(\text{Re}\{h\lambda\} < 0\) => Formula is absolutely stable in all left half \(h\lambda\)-plane

While \(|z_2| < |z_1|\) if \(|h\lambda| < \frac{1}{2}\) formula is relatively stable in the region \(|h\lambda| < \frac{1}{2}\)
F.E.: \(-x_n + x_{n-1} + h \dot{x}_{n-1} = 0 \Rightarrow \dot{x} = \lambda x\)

\[ P(z; h\lambda) = -z + 1 + h\lambda = 0 \]

\[ h\lambda = z - 1 \]

\[
\begin{array}{c|cc}
   \hline
   z   & h\lambda \\
   \hline
   1   & 0 \\
   -1  & -2 \\
   j   & j-1 \\
   0   & -1 \\
   \hline
\end{array}
\]

z-plane

\[ \text{unstable} \]

\[ \text{stable} \]

\[ h\lambda\text{-plane} \]
T.R.: \[-x_n + x_{n-1} + \frac{h}{2} (\dot{x}_n + \dot{x}_{n-1}) = 0\]

\[P(z; \ h\lambda) = -z + 1 + \frac{h\lambda}{2} (z + 1) = 0\]

\[h\lambda = \frac{2(z - 1)}{z + 1}\]

<table>
<thead>
<tr>
<th>z</th>
<th>h\lambda</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>\infty</td>
</tr>
<tr>
<td>j</td>
<td>\frac{2(j - 1)}{j + 1} = 2j</td>
</tr>
<tr>
<td>0</td>
<td>-2</td>
</tr>
</tbody>
</table>

Diagram:
- z-plane
- h\lambda-plane
- Stable region
- Unstable region
- X markers
2nd-Order BDF (or 2nd-Order Gear's Formula) (Constant h)

\[-x_n + \frac{4}{3} x_{n-1} - \frac{1}{3} x_{n-2} + \frac{2}{3} h \dot{x}_n = 0\]

\[P(z; h\lambda) = -z^2 + \frac{4}{3} z - \frac{1}{3} + \frac{2}{3} h \lambda z^2 = 0\]

\[h\lambda = \frac{3z^2 - 4z + 1}{2z^2}\]

<table>
<thead>
<tr>
<th>z</th>
<th>h\lambda</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>j</td>
<td>1 + 2j</td>
</tr>
<tr>
<td>0</td>
<td>\infty</td>
</tr>
</tbody>
</table>

z-plane

\[\text{stable, stable, unstable, unstable}\]

h\lambda-plane
**A-Stability**: A numerical integration formula is A-stable if its region of stability includes the entire left-half plane; i.e., \( \text{Re} \{h\lambda\} < 0 \).

**Examples**: B.E., T.R., 2nd-order BDF with constant coefficients are all A-stable. F.E. is not A-stable.

**Dalquist's Theorem:**
1) An explicit \( \text{Imf} \) cannot be A-stable
2) The order of an A-stable implicit \( \text{Imf} \) cannot exceed two.
3) The 2nd-order A-stable implicit \( \text{Imf} \) with the smallest error constant is the Trapezoidal Rule
Stiffly-Stable Formulas

Formula must be stable when $|h\lambda| \to \infty$ in left-half plane (lhp) (why?)
Consider: \( lmf \) applied to \( \dot{x} = \lambda x \)

\[
P(z; h\lambda) = \sum_{i=0}^{k} \alpha_i z^{k-i} + h\lambda \sum_{i=0}^{k} \beta_i z^{k-i} = 0
\]

\[
= \frac{1}{h\lambda} \left( \sum_{i=0}^{k} \alpha_i z^{k-i} \right) + \left( \beta_0 z^k + \beta_1 z^{k-1} + \ldots + \beta_k \right) = 0
\]

If \(|h\lambda| \to \infty\), \( P(z, \infty) = \beta_0 z^k + \beta_1 z^{k-1} + \ldots + \beta_k = 0 \)

To maintain stability and accuracy when \(|h\lambda| \to \infty\), choose \( \beta_1 = \beta_2 = \ldots = \beta_k = 0 \)
To have the numerical solution → 0 as |hλ| → ∞, we want roots of \( P(z; \infty) \rightarrow 0 \).

Select \( \beta_1 = \beta_2 = \ldots = \beta_k = 0 \)

Then \( P(z, \infty) = \beta_0 z^k = 0 \) and all roots → 0 as |hλ| → ∞.

Get BDFs: \[ \sum_{i=0}^{k} \alpha_i x_{n-i} + h\beta_0 \dot{x}_n = 0 \] which turn out to be stiffly-stable for \( k = 1, 2, \ldots, 6 \).

\( z = -1 \) => not damped at \( \infty \).

Note T.R.: \( P(z; h\lambda) = -z_1 + 1 + \frac{h\lambda}{2} (z + 1) = 0 \)

\( P(z, \infty) = \frac{1}{2} (z + 1) = 0 \)

\( z = -1 \) => not damped at \( \infty \).
BDFs for Gear's Formulas for $p = 1, 2, \ldots, 6$)

$p = 1$: $x_n = x_{n-1} + h \dot{x}_n$ (same as B.E.)

$$\text{LTE} = -\frac{1}{2} h^2 \frac{d^2x_n}{dt^2}$$

(Assume const. $h$):

$p = 2$: $x_n = \frac{4}{3} x_{n-1} - \frac{1}{3} x_{n-2} + \frac{2}{3} h \dot{x}_n$

$$\text{LTE} = -\frac{2}{9} h^3 \frac{d^3x_n}{dt^3}$$

$p = 3$: $x_n = \frac{1}{11} [8x_{n-1} - 9x_{n-2} + 2x_{n-3} + 6h \dot{x}_n]$

$$\text{LTE} = -\frac{3}{22} h^4 \frac{d^4x_n}{dt^4}$$
BDFs for Gear's Formulas for $p = 1, 2, \ldots, 6$ (cont.)

$p = 4$: $x_n = \frac{1}{25} \left[ 48x_{n-1} - 36x_{n-2} + 16x_{n-3} - 3x_{n-4} + 12h \dot{x}_n \right]$

$$LTE = -\frac{12}{125} h^5 \frac{d^5x_n}{dt^5_n}$$

$p = 5$: $x_n = \frac{1}{137} \left[ 300x_{n-1} - 300x_{n-2} + 200x_{n-3} - 75x_{n-4} + 12x_{n-5} + 60h \dot{x}_n \right]$

$$LTE = -\frac{60}{1029} h^6 \frac{d^6x_n}{dt^6_n}$$

$p = 6$: $x_n = \frac{1}{147} \left[ 360x_{n-1} - 450x_{n-2} + 400x_{n-3} - 225x_{n-4} + 72x_{n-5} - 10x_{n-6} + 60h \dot{x}_n \right]$

$$LTE = -\frac{60}{1029} h^7 \frac{d^7x_n}{dt^7_n}$$
Regions of Stability of BDFs
Adam’s Formulas: 

\[-x_n + x_{n-1} + h \sum_{j=0}^{k} \beta_j \dot{x}_{n-j} = 0\]
Exercise 1:

Consider

\[ x_n = \frac{1}{3} (4x_{n-1} - x_{n-2} + 2h \dot{x}_n) \]

\[ k = 2 \]

\[ p: \quad 1 = \frac{1}{3} (4 - 1) = \checkmark \]

\[ t: \quad t_n = \frac{1}{3} (4t_{n-1} - t_{n-2} + 2h) \]

\[ 2h = \frac{1}{3} (4h^2 - 0 + 2h(4h)) \quad \checkmark \]

\[ t^2: \quad t_n^2 = \frac{1}{3} (4t_{n-1}^2 - t_{n-1}^2 + 2h (2t_n)) \]

\[ (2h)^2 = \frac{1}{3} (4h^2 - 0 + 2h(4h)) \quad \checkmark \]

order \( p = 2 \)
Region of Stability: apply to $\dot{x} = \lambda x$

$$P(z, h\lambda) = -z^2 + \frac{4}{3} z - \frac{1}{3} + \frac{2}{3} h \lambda z^2$$

$$h \lambda = \frac{z^2 - \frac{4}{3} z + \frac{1}{3}}{\frac{2}{3} z^2}$$
Exercise 2

Consider

\[x_n = x_{n-1} + \frac{h}{12} (5\dot{x}_n + 8\dot{x}_{n-1} - \dot{x}_{n-2})\]

$k = 2$ (why?)

How to find the order $p$:
Try polynomials $1$, $t$, $t^2$, $t^3$ to determine $p$. 
Region of Stability

\[-z^2 + z + \frac{h\lambda}{12} (5z^2 + 8z - 1) = 0\]

\[h\lambda = \frac{12(z^2 - z)}{5z^2 + 8z - 1}\]

\[
\begin{array}{c|c}
  z & h\lambda \\
  \hline
  1 & 0 \\
  -1 & -6 \\
  j & \frac{6}{25} (-1 - 12j) \\
  8 & \frac{12}{5}
\end{array}
\]

\[\text{stable}\]

\[\text{unstable}\]

\[\text{h}\lambda\text{-plane}\]

\[\text{z-plane}\]
Exercise 3
Consider state equations

\[\begin{align*}
\dot{x}_1 &= -x_1 + x_2 \\
\dot{x}_2 &= x_1 + \alpha x_2, \quad \alpha = +1, -1
\end{align*}\]

Stability of system

\[
\begin{bmatrix} -1 & 1 \\ 1 & \alpha \end{bmatrix} x
\]

Eigenvalues: \(\det [\lambda I - A] = \det \begin{bmatrix} \lambda + 1 & -1 \\ -1 & \lambda - \alpha \end{bmatrix}\)

\[
\det = (\lambda + 1)(\lambda - \alpha) - 1 = 0
\]

\[
\lambda^2 - \alpha \lambda + \lambda - \alpha - 1 = \lambda^2 + (1 - \alpha)\lambda - \alpha - 1 = 0
\]
\[ \lambda = \frac{-(1 - \alpha) \pm \sqrt{(1 - \alpha)^2 + 4(1 + \alpha)}}{2} \]

\( \alpha = 1 \quad \lambda = \frac{\pm \sqrt{8}}{2} = \pm \sqrt{2} \quad \} \text{unstable} \)

\( \alpha = -1 \quad \lambda = -2 \pm \sqrt{2^2 + 4(0)/2} \)

\[ = \frac{2 \pm 2}{2} = -2, 0 \quad \} \text{marginally stable} \]
Numerical Solution

(a) Consider $x_n = \frac{4}{3} x_{n-1} - \frac{1}{3} x_{n-2} + \frac{2}{3} h \dot{x}_n$.

Apply formula to system equations.

$\alpha = 1 \implies \lambda = \pm \sqrt{2}$ unstable

$0 < h\lambda < 4$

$0 < h < \frac{4}{\sqrt{2}} = 2\sqrt{2}$ Inside region of instability

$\alpha = -1, \lambda = 0, -2 \implies h > 0$ stable
(b) Consider

Apply formula to system equations.

\[ x_n = x_{n-1} + \frac{h}{12} (5\dot{x}_n + 8\dot{x}_{n-1} - \dot{x}_{n-2}) \]

\[ \alpha = 1, \lambda = \pm \sqrt{2} \Rightarrow \text{unstable: } h > 0 \]

\[ \alpha = -1 \Rightarrow \lambda = 0, -2 \]

\[ -6 < h(-2) < 0 \Rightarrow 0 < h < 3 \]
Exercise 4

In the following RLC circuit determine the range of the time step \( h \) so that the numerical solution behaves the same way as the analytic solution from the stability point of view. Consider B.E., F.E., and T.R.

\[ L > 0, \ C > 0, \ R = 7, \ 0, \ -7 \ (\text{three cases}) \]
State Equations

Or,

\begin{align*}
C \frac{dv_c}{dt} &= i_L \\
-E(t) + Ri_L + L \frac{di_L}{dt} + v_c &= 0
\end{align*}

Or,

\begin{align*}
L \frac{di_L}{dt} &= -v_c - Ri + E(t) \\
\frac{dv_c}{dt} &= \frac{1}{C} i_L \\
\frac{di_L}{dt} &= \frac{1}{L} (-Ri_L - v_c) + \frac{1}{L} E(t)
\end{align*}
\[
\begin{bmatrix}
\dot{v}_c \\
i_L
\end{bmatrix} = \begin{bmatrix}
0 & \frac{1}{C} \\
-\frac{1}{L} & -\frac{R}{L}
\end{bmatrix} \begin{bmatrix}
v_c \\
i_L
\end{bmatrix} + \begin{bmatrix}
0 \\
1
\end{bmatrix} E(t)
\]

\[A = \begin{bmatrix}
0 & 10 \\
-1 & -R
\end{bmatrix}\]

\[\det [\lambda I - A] = \det \begin{bmatrix}
\lambda & -10 \\
+1 & \lambda + R
\end{bmatrix}\]

\[\lambda^2 + R\lambda + 10 = 0\]

\[\lambda = \frac{-R \pm \sqrt{R^2 - 40}}{2}\]
<table>
<thead>
<tr>
<th>R = 7,</th>
<th>$\lambda = \frac{-7 \pm \sqrt{49 - 40}}{2} = -5, -2$</th>
<th>(stable)</th>
</tr>
</thead>
<tbody>
<tr>
<td>R = 0,</td>
<td>$\lambda = \frac{\sqrt{-40}}{2} = \pm j \sqrt{10}$</td>
<td>(oscillatory)</td>
</tr>
<tr>
<td>R = -7,</td>
<td>$\lambda = (7 \pm \sqrt{49 - 40})/2 = 5, 2$</td>
<td>(unstable)</td>
</tr>
</tbody>
</table>
OR, use Laplace Transform

$$Y(s) = \frac{1}{R + SL + \frac{1}{SC}}$$

$$= \frac{SC}{S^2LC + SRC + 1}$$

Poles

$$s = \frac{-RC \pm \sqrt{(RC)^2 - 4LC}}{2LC}$$

$$s = \frac{-R \pm \sqrt{R^2 - 40}}{2}$$

R = 7, \hspace{1cm} s = \frac{-7 \pm \sqrt{9}}{2} = -5, -2

R = 0, \hspace{1cm} s = \frac{\pm \sqrt{-40}}{2} = \pm j \sqrt{10}

R = -7, \hspace{1cm} s = \frac{+7 \pm \sqrt{9}}{2} = 5, 2$$
Limits on $h$

- $R = 7, \lambda = -5, -2$:
  - B.E. $h > 0$
  - F.E. $h < 2/5$
  - T.R. $h > 0$

- $R = 0, \lambda = -j\sqrt{10}, j\sqrt{10}$:
  - B.E. $h = 0$
  - F.E. $h = 0$
  - T.R. $h > 0$

- $R = -7, \lambda = 5, 2$:
  - B.E. $h < 2/5$
  - F.E. $h > 0$
  - T.R. $h > 0$