Chapter Five

RELAXATION OR ITERATIVE TECHNIQUES FOR THE SOLUTION OF LINEAR EQUATIONS
Vector Norms

\[ \| x \|_p = (| x_1 |^p + | x_2 |^p + \ldots + | x_n |^p)^{\frac{1}{p}} \quad p \geq 1 \]

The 1, 2 and \( \infty \) norms are the most commonly used:

\[ \| x \|_1 = | x_1 | + | x_2 | + \ldots + | x_n | \]

\[ \| x \|_2 = (| x_1 |^2 + | x_2 |^2 + \ldots + | x_n |^2)^{\frac{1}{2}} = (x^T x)^{\frac{1}{2}} \]

\[ \| x \|_\infty = \max_{1 \leq i \leq n} | x_i | \]

\( \| x \|_2 \) is also known as Euclidean norm, and \( \| x \|_1 \) as Manhattan norm.
Properties of Vector Norms

1. $\|x\| > 0$ for all $x \neq 0$.

2. $\|x\| = 0$ iff $x = 0$.

3. $\|\alpha x\| = |\alpha| \|x\|$ for any scalar $\alpha$.

4. $\|x + y\| \leq \|x\| + \|y\|$ for any two vectors $x$ and $y$. 
Matrix Norms

Given $Ax = y$

'Induced' Norm

The norm of a matrix measures the maximum “stretching” the matrix does to any vector in the given vector norm.

$$
\| A \|_1 = \max_{1 \leq j \leq n} \sum_{i=1}^{m} |a_{ij}| \quad \text{(maximum column sum)}
$$

$$
\| A \|_{\infty} = \max_{1 \leq i \leq m} \sum_{j=1}^{n} |a_{ij}| \quad \text{(maximum row sum)}
$$

$$
\| A \|_2 = \left[ \rho(A^T A) \right]^{\frac{1}{2}} \quad \text{(square root of the largest eigenvalue of } A^T A)\)
$$
Properties of Matrix Norms

(1) $\|A\| > 0$ if $A \neq 0$

(2) $\|A\| = 0$ if $A = 0$

(3) $\|\alpha A\| = |\alpha| \cdot \|A\|$ for any scalar $\alpha$

(4) $\|A + B\| \leq \|A\| + \|B\|$

(5) $\|AB\| \leq \|A\| \cdot \|B\|$

(6) $\|Ax\| \leq \|A\| \cdot \|x\|$ for any vector $x$
Relaxation or Iterative Methods of Solving $Ax = b$

- **Aim:** Generate a sequence of vectors $x_0, x_1, ..., x_t$ that will "hopefully" converge to the solution $x^* = A^{-1} b$ (without finding $A^{-1}$ or the LU factors of $A$, just $A$)

- Gauss-Jacobi

- Gauss-Seidel
Point Gauss-Jacobi

\[ \text{Ax} = b \]

\[
\begin{align*}
    a_{11}x_1 & + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
    a_{21}x_1 & + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
    \vdots & \quad \vdots \quad \vdots \\
    a_{n1}x_1 & + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \\
\end{align*}
\]

\[ a_{ii} \neq 0, \ i = 1, \ldots, n \]

Starting with an initial guess \( x^{(0)} \), for \( i = 1, 2, \ldots, n \), solve

\[
x_i^{(k+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1, j \neq i}^{n} a_{ij}x_j^{(k)} \right)
\]

Repeat until \( ||x^{k+1} - x^k|| < \varepsilon \) (can be done in parallel)
Block Gauss-Jacobi

\[ A x = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1p} \\ A_{21} & A_{22} & \cdots & A_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ A_{p1} & A_{p2} & \cdots & A_{pp} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_p \end{bmatrix} \]

Start with initial guess \( x^{(0)} \)

Then solve

\[
A_{ii}x_i^{(k+1)} = b_i - \sum_{j=1}^{n} A_{ij}x_j^{(k)} \quad \text{can be done in parallel}
\]

Repeat until convergence:

\[ \| x^{(k+1)} - x^{(k)} \| < \varepsilon \]
Point Forward Gauss-Seidel

\( k = 0 \), initial guess \( \mathbf{x}^0 \)

\[
\begin{align*}
    x_1^{k+1} &= \frac{1}{a_{11}} \left( b_1 - a_{12}x_2^k - a_{13}x_3^k - \cdots - a_{1n}x_n^k \right) \\
    x_2^{k+1} &= \frac{1}{a_{22}} \left( b_2 - a_{21}x_1^{k+1} - a_{23}x_3^k - \cdots - a_{2n}x_n^k \right)
\end{align*}
\]

\[
x_i^{(k+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k+1)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k)} \right)
\]

Repeat until \( ||\mathbf{x}^{k+1} - \mathbf{x}^k|| < \varepsilon \)
Point Backward Gauss-Seidel

\[ x_n^{(k+1)} = \frac{1}{a_{nn}} \left( b_n - a_{n1}x_1^{(k)} - \cdots - a_{n,n-2}x_{n-2}^{(k)} - a_{n,n-1}x_{n-1}^{(k)} \right) \]

then for \( i = n - 1, \ldots, 1 \), solve

\[ x_i^{(k+1)} = \frac{1}{a_{ii}} \left( b_i - \sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k+1)} \right) \]
Block Forward Gauss-Seidel

- Initial guess $\mathbf{x}^{(0)}$

- Solve
  \[
  A_{ii}x_{i}^{k+1} = b_{i} - \sum_{j=1}^{i-1} A_{ij}x_{j}^{k+1} - \sum_{j=i+1}^{n} A_{ij}x_{j}^{k}
  \]
  \[i = 1, 2, \ldots, p\]

- Repeat until $||\mathbf{x}^{k+1} - \mathbf{x}^{k}|| < \varepsilon$
Block Backward Gauss-Seidel

- Initial guess $x^{(0)}$

- Solve
  \[ A_{ii}x_i^{(k+1)} = b_i - \sum_{j=1}^{i-1} A_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} A_{ij}x_j^{(k+1)} \]

  $i = p-1, p-2, \ldots, 1$

- Repeat until $||x^{k+1} - x^k|| < \epsilon$
Symmetrical Gauss-Seidel Method

• A symmetrical G-S method performs a forward (point or block) G-S iteration followed by a backward G-S iteration
G-J and G-S for bordered-block-diagonal matrices

\[
\begin{bmatrix}
A_{11} & 0 & \cdots & A_{1t} \\
A_{22} & \ddots & \ddots & \ddots \\
\vdots & \ddots & \ddots & \ddots \\
0 & \cdots & A_{pp} & \ddots \\
A_{t1} & A_{t2} & \cdots & A_{tt}
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
\vdots \\
x_t
\end{bmatrix}
=
\begin{bmatrix}
b_1 \\
b_2 \\
\vdots \\
b_t
\end{bmatrix}
\]
G-J and G-S for bordered-block-diagonal matrices

G-J:

\[ A_{ii}x_i^{(k+1)} = b_i - A_{it}x_t^{(k)} \quad i = 1, 2, \ldots, p \]
\[ A_{tt}x_t^{(k+1)} = b_t - \sum_{j=1}^{p} A_{tj}x_j^{(k)} \]

G-S:

\[ A_{ii}x_i^{(k+1)} = b_i - A_{it}x_t^{(k)} \quad i = 1, 2, \ldots, p \]
\[ A_{tt}x_t^{(k+1)} = b_t - \sum_{j=1}^{p} A_{ti}x_j^{(k+1)} \]
Matrix Splitting

More formally,

**Given** \( Ax = b \)

Let \( A = L + D + U \) (matrix splitting)

where \( L \) is lower triangular (strictly)
\( U \) strictly upper triangular
\( D \) point or block-diagonal nonsingular matrix
Gauss-Jacobi

• \( A = L + D + U \)

• Solve \( D x^{k+1} = b - (L+U) x^k \)

• Or, \( x^{k+1} = D^{-1} b - D^{-1}(L+U) x^k \)

\[= D^{-1} b - M_{GJ} x^k\]

• where \( M_{GJ} = D^{-1}(L + U) \) is called the **Gauss-Jacobi Companion Matrix**
Gauss-Seidel

- \( A = L + D + U \)
- Solve \([L + D]x^{k+1} = b - Ux^k\)

or \(x^{k+1} = [L + D]^{-1}b - [L + D]^{-1}Ux^k\)
  = \([L + D]^{-1}b - M_{GS} x^k\)

where \(M_{GS} = [L + D]^{-1}U\) is called the **Gauss-Seidel Companion Matrix**

- If \(D\) is diagonal, then the methods are referred to as point Gauss-Jacobi and point Gauss-Seidel
- If \(D\) is block-diagonal, then they are referred to as block Gauss-Jacobi and block Gauss-Seidel.
Successive Over-Relaxation (S-O-R)  
(to accelerate convergence)

• \(Ax = b \Rightarrow \omega Ax = \omega b, \quad \omega \) is a scalar

• \(\omega (L + D + U)x = \omega b\)

• \(\omega (L + D + U)x + Dx = \omega b + Dx\)

• \(\omega Lx + Dx = \omega b + Dx - \omega Dx - \omega Ux\)

• \(((\omega L + D)x = \omega b + [(1 - \omega)D - \omega U]\)

• \(M_{SOR} = (\omega L + D)^{-1}[\{(1 - \omega)D - \omega U]$$}
• If $D$ is diagonal then

$$a_{ii} x_i^{(k+1)} = \omega b_i - \omega \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} + (1 - \omega) a_{ii} x_i^{(k)} - \omega \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)}$$

$$x_i^{(k+1)} = \frac{\omega}{a_{ii}} \left( b_i - \sum_{j=i+1}^{n} a_{ij} x_j^{(k)} - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} \right) + (1 - \omega) x_i^{(k)}$$
Convergence Theorem

The necessary and sufficient condition for the G-J and G-S iterates to converge to a solution for any initial guess is that the spectral radii of $M_{GJ}$ and $M_{GS}$ are strictly less than 1, i.e., all eigenvalues of $M_{GJ}$ and $M_{GS}$ are inside the unit circle.

Definition: Spectral radius of matrix $M$:

$$\rho(M) \triangleq \max |\lambda_i|$$

eigenvalue of $M$

- **G-J Convergence**: $\rho(M_{GJ}) < 1$
- **G-S Convergence**: $\rho(M_{GS}) < 1$
- **SOR Convergence**: $\rho(M_{SOR}) < 1$
Eigenvalues and Eigenvectors

- Let $M \in \mathbb{R}^{n \times n}$, $\lambda \in \mathbb{C}$ is an **eigenvalue** of $M$ if there exists a $u \in \mathbb{C}^n$, $u \neq 0$, such that
  $$Mu = \lambda u \Rightarrow (M - \lambda I)u = 0; \ u \text{ is an eigenvector},$$

- The eigenvalues can be found by computing the roots of the characteristic polynomial of $M$:
  $$\phi(\lambda) = \det (\lambda I - M) = 0$$
  $$\phi(\lambda)$$ is a polynomial of degree $n$ with real coefficients. It has $n$ (possibly complex) roots.
Eigenvalues and Eigenvectors (cont.)

1. If $\lambda \in \mathbb{R}$, then $u \in \mathbb{R}^n$

2. If $\lambda \in \mathbb{C}$, then $u \in \mathbb{C}^n$, and complex conjugate $\lambda^*$ is also an eigenvalue and $u^*$ is an eigenvector

3. If $A = A^T$ (symmetric matrix), then all $\lambda$’s are real, and all $u \in \mathbb{R}^n$.

4. The spectral radius of $M$ is $\rho(M) = \max |\lambda_i|$

5. $\rho(M) \leq ||M||$, i.e., $0 \leq \rho(M) \leq ||M||$

6. For symmetric matrices, $\rho(M) = ||M||_2$
Eigenvalues and Eigenvectors (cont.)

- **Lemma 1** If \( \rho(M) < 1 \), then \( \lim_{k \to \infty} M^k = \varphi_{n \times n} \), where
  \[ M^k = M \cdot M \cdot M \ldots \]
  \( \varphi_{n \times n} \) is an \( n \times n \) zero matrix

- **Lemma 2** If \( \rho(M) < 1 \), then
  \( (I - M)^{-1} \) exists, and
  \( (I - M)^{-1} = I + M + M^2 + \ldots \)

  **Scalar case, \( a \):** \( \rho(a) = |a| \)

  - if \( |a| < 1 \), then \( \lim a^k = 0 \) when \( k \to \infty \)
  - \( 1/(1-a) = (1-a)^{-1} = 1 + a + a^2 + \ldots \)
Convergence Theorem

- Given $x = Mx + c$ with solution $x^*$, then $(I - M)^{-1}$ exists and $x^* = (I - M)^{-1}c$

Let $x^{k+1} = Mx^k + c$, then $\lim_{k \to \infty} x^k \to x^*$ iff $\rho(M) < 1$

Proof:

$x^{(1)} = M x^{(0)} + c$

$x^{(2)} = M x^1 + c = M(M x^0 + c) + c$

$\vdots = M^2 x^0 + Mc + c$

$x^{(k)} = M^k x^0 + (I + M + M^2 + \ldots + M^{k-1}) c$

Since $\rho(M) < 1$, $\lim_{k \to \infty} M^k = \Phi$, and $(I + M + M^2 + \ldots + M^{k-1}) \to (I - M)^{-1}$

$\therefore x^k \to (I - M)^{-1} c = x^*$ as $k \to \infty$
Given:  \( Ax = b, \quad A = L + D + U \)

**G-J:**  \( x^{k+1} = D^{-1}b - D^{-1}(L+U)x^k \)
\[ M_{GJ} = D^{-1}(L+U) \]

**G-S:**  \( x^{k+1} = (L+D)^{-1}b - (L+D)^{-1}Ux^k \)
\[ M_{GS} = (L+D)^{-1}U \]

**SOR:**  \( (\omega L + D)x^{k+1} = \omega b + [(1 - \omega)D - \omega U]x^k \)
\[ M_{SOR} = (\omega L + D)^{-1}[(1 - \omega)D - \omega U] \]
Convergence

• Necessary and Sufficient condition for convergence:
  \[ \rho(M_{GJ}) < 1, \rho(M_{GS}) < 1 \]

However, \( \rho(M) \leq ||M|| \)

• Sufficient (but not necessary) condition for convergence:
  If \( ||M|| < 1 \) for some induced norm, then \( x^{k+1} = M x^k + c \) converges to the solution \( x^* \).

• Proof: \( \rho(M) < ||M|| < 1 \)

In practice, use \( ||M||_1 \) (maximum column sum)
  or \( ||M||_\infty \) (maximum row sum)
Convergence test using the original matrix $A$
(sufficient conditions)

Given $Ax = b$

- *Regular* splitting: $A = M - N$. $M$ nonsingular, $M^{-1} > 0$, $N > 0$.

- M-matrix: Nonsingular, $a_{ii} > 0$, $a_{ij} \leq 0$, $i \neq j \implies A^{-1} \geq 0$.

- The admittance matrix of a linear resistive circuit with no controlled sources and no voltage sources is an M-matrix.

- If $A$ is an M-matrix, then the GJ and GS iterative methods converge for any initial point $x_0$. 
Convergence test using the original matrix $A$
(sufficient conditions)

- $A$ is diagonally dominant (d.d.) if

\[ \left| a_{ii} \right| \geq \sum_{j=1, j \neq i}^{n} \left| a_{ij} \right| \]

- $A$ is strictly diagonally dominant (s.d.d.) if

\[ \left| a_{ii} \right| > \sum_{j=1, j \neq i}^{n} \left| a_{ij} \right| \]
Convergence test using the original matrix $A$ (sufficient conditions)

- **Irreducible**: Let $G(A)$ be the directed graph of $A$. If $G(A)$ is strongly-connected, then $A$ is irreducible.

- **Strongly-connected**: For every pair of distinct vertices $i$ and $j$ in $G(A)$, there exists a directed path from $i$ to $j$ and from $j$ to $i$.

![Diagram](image)  
Not strongly-connected
Convergence test using the original matrix A (sufficient conditions)

- If $A$ is reducible, then order $A$ can be reordered to have **leading top right-hand zeros**:
Convergence test using the original matrix A
sufficient conditions

• A is irreducibly diagonally dominant (i.d.d.) if it is irreducible, diagonally dominant, with at least one row strictly diagonally dominant.
Theorem

• If $A$ is s.d.d. (does not have to be irreducible), or i.d.d., then $\rho (M_{GJ}) < 1$ and $\rho (M_{GS}) < 1$; and point G-J, block G-J, point G-S, and block G-S, converge to the solution of $Ax = b$.

• Remark:

  *Diagonal dominance* and, in general, the *convergence* of G-J and G-S depend on *matrix ordering*. Convergence block G-J and block G-S also depends on *matrix partitioning*.
Convergence test using the original matrix A
sufficient conditions

• **A** is **positive definite** if \( u^T A u > 0 \) for all vectors \( u \neq 0 \)

• A **symmetric** positive definite matrix has real, positive eigenvalues

• If **A** is symmetric and diagonally dominant with **positive** diagonal elements, the **A** is positive definite

• If **A** is symmetric and positive definite then G-J and G-S converge
Convergence test for SOR methods

\[(\omega L + D) x = \omega b + [(1 - \omega)D - \omega U]\]

- For a few structured problems, \(\omega\) is determined by minimizing \(\rho(M^{-1}N)\), where \(M = (\omega L + D), \ N = [(1 - \omega)D - \omega U]\)
- In general \(\omega\) may be expensive to find. It is determined from “experience” in solving a certain class of problems.
- If \(A\) is symmetric and positive definite then the SOR method converges for any initial guess and for \(0 < \omega < 2\)
Example

Ax = b

1. \[ A = \begin{bmatrix} 5 & 2 \\ 4 & 4 \end{bmatrix} \]; A is i.d.d., G-J and G-S will converge

2. \[ A = \begin{bmatrix} 4 & 4 \\ 5 & 2 \end{bmatrix} \] not d.d. Need to do additional tests.

\[
G-J \quad M_{GI} = D^{-1} (L + U)
\]

\[
= \begin{bmatrix} 4 & 2 \\ 2 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix}
\]

\[
= \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 5 & 0 \end{bmatrix}
\]
Example (cont.)

\[
\begin{bmatrix}
0 & 1 \\
5/2 & 0
\end{bmatrix}
\]

\[
M_{GJ} = \begin{bmatrix}
0 & 1 \\
5/2 & 0
\end{bmatrix}; \quad \|M_{GJ}\| = 5/2. \quad \text{Need to find eigenvalues.}
\]

\[
\text{det}\{\lambda I - M_{GJ}\} = \text{det} \begin{bmatrix}
\lambda & -1 \\
-5/2 & \lambda
\end{bmatrix}
\]

\[
\lambda^2 - 5/2 = 0 \Rightarrow |\lambda| = \frac{\sqrt{5}}{2} > 1 \quad \Rightarrow \quad \text{Divergence}
\]
Example

$$Ax = b$$

$$A = \begin{bmatrix} 4 & 4 \\ 5 & 2 \end{bmatrix}$$

**G-S**

$$M_{GS} = \begin{bmatrix} 4 & 0 \\ 5 & 2 \end{bmatrix}^{-1} \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}$$
Convergence checks (cont.)

\[
M_{GS} = \frac{1}{8} \begin{bmatrix} 2 & 0 \\ -5 & 4 \end{bmatrix} \begin{bmatrix} 0 & 4 \\ 0 & 0 \end{bmatrix}
\]

\[
= \frac{1}{8} \begin{bmatrix} 0 & 8 \\ 0 & -20 \end{bmatrix}
\]

\[
\rho(M) = \frac{20}{8} > 1
\]

\[
= \begin{bmatrix} 0 & \frac{1}{8} \\ 0 & \frac{-20}{8} \end{bmatrix}
\]

\[
\Rightarrow \quad \lambda = 0, \quad \frac{-20}{8} \quad \Rightarrow \quad \text{Divergence}
\]
Convergence checks (cont.)

\[
A = \begin{bmatrix}
1 & 2 & -2 \\
1 & 1 & 1 \\
2 & 2 & 1
\end{bmatrix}
\]

G-J: \( M_{GJ} = \begin{bmatrix}
1 & \quad & \\
\quad & 1 & \\
\quad & \quad & 1
\end{bmatrix}^{-1} \begin{bmatrix}
0 & 2 & -2 \\
1 & 0 & 1 \\
2 & 2 & 0
\end{bmatrix} = \begin{bmatrix}
0 & 2 & -2 \\
1 & 0 & 1 \\
2 & 2 & 0
\end{bmatrix} \Rightarrow \|M\| = 4
Convergence checks (cont.)

\[
\det (\lambda I - M_{GJ}) = \det \begin{bmatrix}
\lambda & -2 & 2 \\
-1 & \lambda & -1 \\
-2 & -2 & \lambda
\end{bmatrix} = \lambda^3 + 4 - 4 + 4\lambda - 2\lambda - 2\lambda
\]

\[
\det = \lambda^3 + 4 - 4 + 4\lambda - 2\lambda - 2\lambda = \lambda^3 = 0 \implies \lambda = 0 \quad \text{Convergence}
\]

G-S: \[M_{GS} = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
2 & 2 & 1
\end{bmatrix} (L+D)^{-1} \begin{bmatrix}
0 & 2 & -2 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
M_{GS} = \begin{bmatrix}
1 & 0 & 0 \\
1 & 1 & 0 \\
2 & 2 & 1
\end{bmatrix}
\begin{bmatrix}
0 & 2 & -2 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]
Convergence checks (cont.)

\[
M_{GS} = \begin{bmatrix}
1 & 0 & 0 & 0 & 2 & -2 \\
1 & 0 & 0 & -2 & (1+2) \\
1 & 0 & -4+4 & +4-6
\end{bmatrix}
\]

\[
= \begin{bmatrix}
0 & 2 & -2 \\
0 & -2 & 3 \\
0 & 0 & -2
\end{bmatrix} \Rightarrow \lambda = 0, -2, -2 \Rightarrow \text{Divergence}
\]
Block G-J and Block G-S (Examples)

\[
A = \begin{bmatrix}
1 & 2 & -2 \\
1 & 1 & 1 \\
2 & 2 & 1 \\
\end{bmatrix}
\]

Block G-J: \( M_{GJ} = \begin{bmatrix}
1 & 2 & 0 \\
1 & 1 & 1 \\
0 & 1 & 1 \\
\end{bmatrix} \begin{bmatrix}
0 & -2 \\
\end{bmatrix}
\]

\[
M_{GJ} = \begin{bmatrix}
-1 & 2 & 0 \\
1 & -1 & 1 \\
0 & 2 & 0 \\
\end{bmatrix} \begin{bmatrix}
0 & -2 \\
1 & 1 \\
\end{bmatrix} = \begin{bmatrix}
0 & 0 & 4 \\
0 & 0 & -3 \\
2 & 2 & 0 \\
\end{bmatrix};
\]

\[
\det (\lambda I - M_{GJ}) = \det \begin{bmatrix}
\lambda & 0 & -4 \\
0 & \lambda & +3 \\
-2 & -2 & \lambda \\
\end{bmatrix}
\]

\[
= \lambda^3 - 8\lambda + 6\lambda
\]

\[
= \lambda(\lambda^2 - 2) \Rightarrow \lambda = 0, \lambda^2 = 2
\]
Block G-J and Block G-S (Examples) (cont.)

\[
M_{GS} = \begin{bmatrix}
1 & 2 & 0 \\
1 & 1 & 0 \\
2 & 2 & 1
\end{bmatrix}^{-1} \begin{bmatrix}
0 & 0 & -2 \\
0 & 0 & 1 \\
0 & 0 & 0
\end{bmatrix}
\]

\[
M_{GS} = \begin{bmatrix}
0 & 0 & 4 \\
0 & 0 & -3 \\
0 & 0 & -2
\end{bmatrix}, \lambda = 0, 0, -2 \quad \text{diverges}
\]

How to check if the roots of a polynomial are within the unit circle without finding the roots (related to stability of linear time-invariant discrete systems)

Use Bilinear Transformation and Routh-Hurwitz Test (R-H)
For **bilinear transformation**, see


For **Routh-Hurwitz Test**, see

Example 1

Given: \(4x^2 + 5x + 3 = 0\). Are all the roots within the unit circle?

Put \(x = \frac{1+z}{1-z}\) bilinear transformation

\[4\left(\frac{1+z}{1-z}\right)^2 + 5\left(\frac{1+z}{1-z}\right) + 3 = 0\]

\[
\frac{4(1+z)^2 + 5(1+z)(1-z) + 3(1-z)^2}{(1-z)^2} = 0
\]

\[4(1+2z+z^2)+5(1-z^2)+3(1-z)^2 = 0\]

\[2z^2 + 2z + 12 = 0\] (R–H test):

* for 2nd order polynomial if all coefficients are of the same sign and none is zero, then all roots are in lhp.
Example 2

Given: \(4x^3 - 4x^2 - 7x - 3 = 0\), Are all the roots within the unit circle?

\[
\text{bilinear transformation} \quad x = \frac{1+z}{1-z}
\]

\[
4 \left( \frac{1+z}{1-z} \right)^3 - 4 \left( \frac{1+z}{1-z} \right)^2 - 7 \left( \frac{1+z}{1-z} \right) - 3 = 0
\]

\[
\frac{4z^2 + 14z^2 + 24z - 10}{(1-z)^3} = 0
\]

\(4z^3 + 14z^2 + 24z - 10 = 0\): Different signs of coeff: some roots in rhp
Another Example

\[ z^3 + z^2 + 2z + 1 = 0 \]

R-H Test

\[ \begin{array}{cc}
1 & 2 \\
1 & 1 \\
1 & 0 \\
\end{array} \]

all roots in lhp
SUMMARY

• To check the convergence of solving $Ax = b$ by G-J or G-S iterative methods:

• Check if $A$ is s.d.d. or i.d.d.

  If it is => Convergence

• If $A$ is reducible, then order $A$ to have leading top right-hand zeros:

![Diagram of matrix with leading top right-hand zeros]
Remark

The matrix representing the linearized circuit equations of a digital combinational circuit (with no feedback), with “simple” transistor models (no gate-to-drain or gate-to-source capacitance) is reducible and can be ordered as a lower block triangular form.
• For each irreducible submatrix (if it is to be solved by \(G-J\) or \(GS\)), construct the companion matrix \(M\), (if the dimension of \(M\) is small) where

\[
M_{GJ} = D^{-1}(L + U)
\]

\[
M_{GS} = (L + D)^{-1}U
\]

• Find eigenvalues of \(M\): Roots of \(\phi(\lambda) = \det(\lambda I - M) = 0\)

• If order of \(\phi(\lambda)\) is three or more and hard to factorize, check if the \textit{roots} of \(\phi(\lambda)\) are strictly inside the unit circle in the complex plane: by applying \textit{bilinear transformation} and the \textit{Routh-Hurwitz test} on the transformed equation.
Finding the eigenvalues, or checking if they are within the unit circle, is very expensive for large matrices, so it is rarely used. Convergence can be checked by checking if the norm of the difference between successive iterations is getting smaller:

\[ ||x^{k+1} - x^k|| < ||x^k - x^{k-1}|| < ||x^{k-1} - x^{k-2}|| \ldots \]
Steepest Descent Method

Given $Ax = b$, $A$ positive definite and symmetric

Error or Residual vector at iteration point $x_k$: $r_k = Ax_k - b$

Define

$$\phi(x) = \frac{1}{2}x^TAx - x^Tb$$

The gradient of $\phi(x)$ is $\nabla \phi(x) = Ax - b$

The minimum of $\phi(x)$ occurs when

$\nabla \phi(x) = 0$; that is, when $Ax - b = 0$.

$$\phi(x_k + \alpha_k r_k) < \phi(x_k)$$
Steepest Descent Method

\[ \phi(x_k + \alpha_k r_k) = \frac{1}{2} x_k^T A x_k + \alpha_k r_k^T A x_k + \frac{1}{2} \alpha_k r_k^T A r_k - x_k^T b - \alpha_k r_k^T b \]

\[ = \phi(x_k) - \alpha_k r_k^T r_k + \frac{1}{2} \alpha_k^2 r_k^T A r_k \]  \hspace{1cm} (6.27)

The value of \( \alpha_k \) that minimizes \( \phi(x_k + \alpha_k r_k) \) along the line \( (x_k + \alpha_k r_k) \) is found by equating to zero the derivative of (6.27) with respect to \( \alpha_k \):

\[ \frac{d}{d\alpha_k} (\phi(x_k + \alpha_k r_k)) = -r_k^T r_k + \alpha_k r_k^T A r_k = 0 \]  \hspace{1cm} (6.28)

or

\[ \alpha_k = \frac{r_k^T r_k}{r_k^T A r_k} \]  \hspace{1cm} (6.29)
Steepest Descent Algorithm

\[ x_0 = \text{initial guess} \]

\[ r_0 = b - A x_0 \]

\[ k = 0 \]

While \( r_k \neq 0 \)

\[ k = k + 1 \]

\[ \alpha_k = r_{k-1}^T r_{k-1} / r_{k-1}^T A r_{k-1} \]

\[ x_k = x_{k-1} + \alpha_k r_{k-1} \]

\[ r_k = b - A x_k \]

end
• Steepest Descent is slow

• Better to choose direction \( p_k \), not necessarily \( r_k \), such that \( \phi(x_{k-1} + \alpha p_k) \) is minimized. This occurs when

\[
\alpha_k = \frac{p_k^T r_{k-1}}{p_k^T A p_k}
\]

• \( p_k \) should not be orthogonal to \( r_{k-1} \)

• if \( p_k \) is orthogonal to \( r_{k-1} \), then \( p_k^T r_{k-1} = 0 \)
Algorithm

\( x_0 \) = initial guess
\( r_0 = b - A x_0 \)
\( k = 0 \)
while \( r_k \neq 0 \)
\( \quad k = k+1 \)
choose a direction \( p_k \) such that \( p_k^T r_{k-1} \neq 0 \)
\( \alpha_k = \frac{p_k^T r_{k-1}}{p_k^T A p_k} \)
\( x_k = x_{k-1} + \alpha_k p_k \)
\( r_k = b - A x_k \)
End
The search directions are linearly independent and $x_k$ solves the problem Min $\mathcal{O}(x)$

Where $x_k = x_0 + \alpha_1 p_1 + \alpha_2 p_2 + \ldots + \alpha_k p_k$

Or $x_k = x_0 + \text{span} \{p_1, p_2, \ldots, p_k\}$

Convergence is guaranteed in at most $n$ steps
Conjugate Gradient Algorithm

1. $x_0 =$ initial guess; compute $r_0 = b - Ax_0$, $p_0 = r_0$
2. For $i = 1, 2, \ldots$, until convergence, Do
3. \[ q_i = Ap_i \]
4. \[ \alpha_i = (r_i, r_i)/(q_i, p_i) \]
5. \[ r_{i+1} = r_i - \alpha_i q_i \]
6. If $\| r_{i+1} \| < \varepsilon$, go to 10
7. \[ x_{i+1} = x_i + \alpha_i p_i \]
8. \[ \beta_i = (r_{i+1}, r_{i+1})/(r_i, r_i) \]
9. \[ p_{i+1} = r_{i+1} + \beta_i p_i \]
10. End
11. $x_{i+1} = x_i + \alpha_i p_i \{ \text{solution} \}$
Other Iterative methods

- Projection methods
- Incomplete LU factorization (ILU)
- Multigrid methods