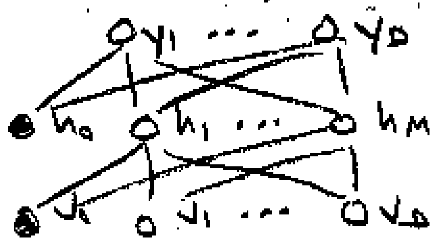
ECE544NA
10/31/2013

TODAY
$\boxed{\text{AUTOENCODERS AND RESTRICTED BOLTZMANN MACHINES}}$

<u>NOTATION</u> $\vec{v} = \vec{v}^{(n)} = [v_1, \ldots, v_D]^T = n^{TH}$ VISIBLE VECTOR

$\vec{h} = \vec{h}^{(n)} = [h_1, \ldots, h_M]^T = n^{TH}$ HIDDEN VECTOR



AUTOENCODER

<u>GOAL</u> : MINIMIZE $E[\|\vec{y} - \vec{v}\|^2]$

$\Rightarrow y_i = E[v_i | \vec{h}]$

SUBJECT TO $y_i = g\left(b_i + \sum_{j=1}^{M} w_{ji} h_j\right) = g(b_i + \vec{w}_i^T \vec{h}$

$h_j = \tilde{g}\left(c_j + \sum_{i=1}^{D} w_{ji} v_i\right) = \tilde{g}(c_j + \vec{m}_j^T \vec{v}$

$$W = \begin{bmatrix} w_{11} & \cdots & w_{1D} \\ \vdots & & \\ w_{M1} & \cdots & w_{MD} \end{bmatrix} = [\vec{w}_1, \ldots, \vec{w}_D] = \begin{bmatrix} \vec{m}_1 \\ \vdots \\ \vec{m}_M \end{bmatrix}$$

THUS $\vec{y} = g(\vec{b} + W^T \vec{h})$, $\vec{h} = \tilde{g}(\vec{c} + W\vec{v})$

<u>GOAL</u> $\vec{y} = \arg\min [\|\vec{y} - \vec{v}\|^2] = \arg\min [\|\vec{y}\|^2 + \|\vec{v}\|^2 - 2\vec{v}^T(\vec{b} +$

$\Rightarrow y_i = E[v_i | \vec{h}]$

<u>BINARY DATA</u> $v_i \in \{0, 1\}$

$y_i = \Pr\{v_i = 1 | \vec{h}\} = g(b_i + \vec{w}_i^T \vec{h})$

CHOOSE $g(a) = \dfrac{1}{1 + e^{-a}}$

$$\Pr\{v_i = 1 \mid \vec{h}\} = \frac{1}{1 + e^{-b_i - \vec{w}_i^T \vec{h}}} = \frac{e^{b_i + \vec{w}_i^T \vec{h}}}{e^{b_i + \vec{w}_i^T \vec{h}} + 1}$$

$$\Pr\{v_i = 0 \mid \vec{h}\} = \frac{e^{-b_i - \vec{w}_i^T \vec{h}}}{1 + e^{-b_i - \vec{w}_i^T \vec{h}}} = \frac{1}{e^{b_i + \vec{w}_i^T \vec{h}} + 1}$$

SO $$\Pr\{v_i \mid \vec{h}\} = \frac{e^{-b_i v_i + v_i \vec{w}_i^T \vec{h}}}{Z} = \frac{e^{-E(v_i, \vec{h})}}{Z}$$

$$Z = \sum_{v_i = 0}^{1} e^{-E(v_i, \vec{h})} \qquad E(v_i, \vec{h}) = -b_i v_i - v_i \vec{w}_i^T \vec{h}$$

SIMILARLY

$$\Pr\{\vec{v} \mid \vec{h}\} = \prod_{i=1}^{D} \Pr\{v_i \mid \vec{h}\} = \frac{e^{-E(\vec{v}, \vec{h})}}{Z(\vec{h})}$$

$$Z(\vec{h}) = \sum_{\vec{v}} e^{-E(\vec{v}, \vec{h})}, \quad E(\vec{v}, \vec{h}) = -\vec{b}^T \vec{v} - \vec{v} W^T \vec{h}$$

TO GET $y_i = E[v_i \mid \vec{h}] = \Pr\{v_i = 1 \mid \vec{h}\}$

WE CAN CHOOSE

$$W = \operatorname{argmax} \prod_{i=1}^{D} \left(\prod_{\substack{n=1 \\ v_i^{(n)} = 1}}^{N} \Pr\{v_i^{(n)} = 1 \mid \vec{h}^{(n)}\} \right) \left(\prod_{\substack{n=1 \\ v_i^{(n)} = 0}}^{N} \Pr\{v_i^{(n)} = 0 \mid \vec{h}^{(n)}\} \right)$$

$$\boxed{W = \operatorname{argmax} \prod_{n=1}^{N} \frac{e^{-E(\vec{v}^{(n)}, \vec{h}^{(n)})}}{Z(\vec{h}^{(n)})}}$$

REAL-VALUED DATA : $v_i \in \mathbb{R}$

$$\vec{y} = g(\vec{b} + W^T\vec{h})$$

Let's use $g(\vec{a}) = \vec{a}$ so

$$\vec{y} = \vec{b} + W^T\vec{h}$$

GOAL: $\vec{b}+W^T\vec{h}$ argmin $E\left[\|\vec{y} - \vec{v}\|^2\right]$

$\Leftrightarrow \vec{b}+W^T\vec{h} = E[\vec{v} \mid \vec{h}]$

SOLUTION: MINIMIZE $E\left[\|\vec{v} - (\vec{b}+W^T\vec{h})\|^2\right]$

BY MAXIMIZING $\prod_{n=1}^{N} p(\vec{v} \mid \vec{h})$

FOR $p(\vec{v}^n \mid \vec{h}^n) \propto \dfrac{e^{-\frac{1}{2}\|\vec{v}^{(n)} - (\vec{b}+W^T\vec{h})\|^2}}{Z}$

WITH GAUSSIAN $\mu = \vec{b}+W^T\vec{h}$, $\Sigma = I$

$$Z = \int e^{-\frac{1}{2}\|\vec{v} - (\vec{b}+W^T\vec{h})\|^2} d\vec{v} = (2\pi)^{D/2}$$

$\boxed{\text{GAUSSIAN}}$

$$p(\vec{v} \mid \vec{h}) = \frac{e^{-E(\vec{v},\vec{h})}}{Z} \qquad Z = \int e^{-E(\vec{v},\vec{h})} d\vec{v} = (2\pi)^{D/2}$$

$$E(\vec{v},\vec{h}) = \frac{1}{2}\|v\|^2 + \frac{1}{2}\|b+W^T\vec{h}\|^2 - \vec{v}^T\vec{b} - \vec{v}^T W^T\vec{h}$$

$\boxed{\text{BINARY}}$

SAME, BUT $E(\vec{v},\vec{h}) = -\vec{v}^T\vec{b} - \vec{v}^T W^T\vec{h}$

$\Leftrightarrow$ ASSUME $\|\vec{v}\|^2$ AND $\|\vec{b}+W^T\vec{h}\|^2$ IRRELEVANT

SYMMETRY

GOAL: $g(W\vec{v} + \vec{c}) = E[\vec{h} \mid \vec{v}]$

$$\Rightarrow \boxed{\begin{aligned} & W, \vec{c} = \arg\max \prod_{n=1}^{N} p\left(\vec{h}^{(n)} \mid \vec{v}^{(n)}\right) \\ & p\left(\vec{h}^{(n)} \mid \vec{v}^{(n)}\right) = \frac{e^{-E(\vec{h}, \vec{v})}}{Z} \\ & Z = \sum_{\vec{h}} e^{-E(\vec{h}, \vec{v})} \\ & E(\vec{h}, \vec{v}) = \begin{cases} -\vec{c}^T \vec{h} - \vec{h}^T W \vec{v} & \vec{h} \in \text{BINARY} \\ \frac{1}{2}\|\vec{h}\|^2 + \frac{1}{2}\|W\vec{v} + \vec{c}\|^2 - \vec{c}^T \vec{h} - \vec{h}^T W \vec{v} & \vec{h} \in \text{REAL} \end{cases} \end{aligned}}$$

BOLTZMANN MACHINE = A NEURAL NET TRAINED AS $(W, \vec{b}) = \arg\max \prod_{n=1}^{N} p(\vec{x}^{(n)})$

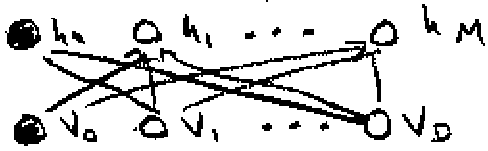
FOR $p(\vec{x}) = \frac{1}{Z} e^{-E(\vec{x})}$, $Z = \sum_{\vec{x}} e^{-E(\vec{x})}$

$E(\vec{x}) = -\vec{x}^T W \vec{x} - \vec{b}^T \vec{x}$ IS CALLED THE "ENERGY"

RESTRICTED BOLTZMANN MACHINE : $\vec{x} = \begin{bmatrix} \vec{h} \\ \vec{v} \end{bmatrix}$ SUCH THAT

$\vec{h} = W\vec{v} + \vec{b}$

$\vec{v} = W^T \vec{h} + \vec{c}$



$(W, \vec{b}, \vec{c}) = \arg\max \prod_{n=1}^{N} p(\vec{x}^{(n)})$

$p(\vec{x}) = \frac{1}{Z} e^{-E(\vec{h}, \vec{v})}$, $E(\vec{h}, \vec{v}) = -\vec{b}^T \vec{v} - \vec{c}^T \vec{h} - \vec{h}^T W \vec{v}$

CONTRASTIVE DIVERGENCE: FOR EACH: n,

① SAMPLE: RANDOMLY CHOOSE A BINARY VECTOR $\vec{h}^{(n)}$ ACCORDING TO $p(\vec{h}^{(n)}|\vec{v}^{(n)}) = \frac{1}{Z} e^{-E(\vec{h},\vec{v}^{(n)})}$

② UPDATE PARAMETERS:

$$w_{ji} \leftarrow w_{ji} + \eta \frac{\partial \ln p(\vec{h}^{(n)}|\vec{v}^{(n)})}{\partial w_{ji}}$$

$$w_{ji} \leftarrow w_{ji} + \eta \frac{\partial \ln p(\vec{v}^{(n)}|\vec{h}^{(n)})}{\partial w_{ji}}$$

$$c_j \leftarrow c_j + \eta \frac{\partial \ln p(\vec{h}^{(n)}|\vec{v}^{(n)})}{\partial c_j}$$

$$b_i \leftarrow b_i + \eta \frac{\partial \ln p(\vec{v}^{(n)}|\vec{h}^{(n)})}{\partial b_i}$$

WHERE $\ln p(\vec{h}|\vec{v}) = \ln \frac{e^{-E(\vec{h},\vec{v})}}{Z}$

$$\frac{\partial \ln p(\vec{h}|\vec{v})}{\partial w_{ji}} = \frac{1}{p(\vec{h}|\vec{v})} \frac{\partial}{\partial w_{ji}} p(\vec{h}|\vec{v})$$

$$= \cancel{\frac{1}{p(\vec{h}|\vec{v})}} \left[\cdot \cancel{\frac{e^{-E(\vec{h},\vec{v})}}{Z}} \left(- \frac{\partial E(\vec{h},\vec{v})}{\partial w_{ji}} \right) - \frac{\cancel{e^{-E(\vec{h},\vec{v})}}}{Z^{\cancel{2}}} \frac{\partial Z}{\partial w_{ji}} \right.$$

$$= -\frac{\partial E(\vec{h},\vec{v})}{\partial w_{ji}} - \frac{\frac{\partial}{\partial w_{ji}} \left(\sum_{\tilde{h}} e^{-E(\tilde{h},\vec{v})} \right)}{Z}$$

$$= -\frac{\partial E(\vec{h},\vec{v})}{\partial w_{ji}} + \frac{1}{Z} \sum_{\tilde{h}} e^{-E(\tilde{h},\vec{v})} \frac{\partial E(\tilde{h},\vec{v})}{\partial w_{ji}}$$

$$= -\frac{\partial E(\vec{h},\vec{v})}{\partial w_{ji}} + \sum_{\tilde{h}} p(\tilde{h}|\vec{v}) \frac{\partial E(\tilde{h},\vec{v})}{\partial w_{ji}}$$

FOR EXAMPLE IF

$$E(\vec{h},\vec{v}) = -\vec{b}^T\vec{v} - \vec{c}^T\vec{h} - \vec{h}^T W \vec{v}$$

THEN
$$\frac{\partial E(\vec{h},\vec{v})}{\partial w_{ji}} = -h_j v_i$$

$$\frac{\partial E(\tilde{h},\vec{v})}{\partial w_{ji}} = -\tilde{h}_j v_i$$

$$\frac{\partial \ln p(\vec{h}|\vec{v})}{\partial w_{ji}} = h_j v_i - \sum_{\tilde{h}_j} p(\tilde{h}_j|\vec{v})\,\tilde{h}_j v_i$$

$$w_{ji} \longleftarrow w_{ji} + \eta\left(h_j v_i - \sum_{\tilde{h}_j} p(\tilde{h}_j|\vec{v})\,\tilde{h}_j v_i\right)$$

$$w_{ji} \longleftarrow w_{ji} + \eta\left(h_j v_i - \sum_{\tilde{v}_i} p(\tilde{v}_i|\vec{h})\,h_j \tilde{v}_i\right)$$

$$b_i \longleftarrow b_i + \eta\left(v_i - \sum_{\tilde{v}_i} p(\tilde{v}_i|\vec{h})\,\tilde{v}_i\right)$$

$$c_j \longleftarrow c_j + \eta\left(h_j - \sum_{\tilde{h}_j} p(\tilde{h}_j|\vec{v})\,\tilde{h}_j\right)$$